

# Einstein-Hilbert action with cosmological term from Chern-Simons gravity

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## Abstract

We propose a modification to the Lie algebra  $S$ -expansion method. The modification is carried out by imposing a condition on the  $S$ -expansion procedure, when the semigroup is given by a cyclic group of even order. The  $S$ -expanded algebras are called  $S_H$ -expanded algebras where  $S = Z_{2n}$ . The invariant tensors for  $S_H$ -expanded algebras are calculated and the dual formulation of  $S_H$ -expansion procedure is proposed. We consider the  $S_H$ -expansion of the five-dimensional  $AdS$  algebra and its corresponding invariant tensors are found. Then a Chern-Simons Lagrangian invariant under the five-dimensional  $AdS$  algebra  $S_H$ -expanded is constructed and its relationship to the general relativity is studied. <sup>†</sup>

## 1 Introduction

The Lie algebra expansion procedure was introduced for the first time in Ref. [1], and subsequently studied in general in Refs. [2], [3], [4]. The expansion procedure is different from the Inönü-Wigner contraction method [5] albeit, when the algebra dimension does not change in the process, it may lead to a simple Inönü-Wigner or generalized Inönü-Wigner contraction in the sense of Weimar-Woods [6], [7], [8]. Furthermore, the algebras to which it leads have in general higher dimension than the original one, in which case they cannot be related to it by any contraction or deformation process.

The expansion method proposed in Refs. [1], [2] consists in considering the original algebra as described by its associated Maurer-Cartan forms on the group manifold. Some of the group parameters are rescaled by a factor  $\lambda$ , and the Maurer-Cartan forms are expanded as a power series in  $\lambda$ . This series is finally truncated in a way that assures the closure of the expanded algebra.

In Refs. [9], [10], [11] was proposed a natural outgrowth of power series expansion method, which is based on combining the structure constant of the

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algebra  $\mathfrak{g}$  with the inner law of a semigroup  $S$  in order to define the Lie bracket of a new  $S$ -expanded algebra.

Theorem 1 of Ref. [2] shows that, in the more general case, the expanded Lie algebra has the structure constants

$$C_{(A,i)(B,j)}^{(C,k)} = \begin{cases} 0 & \text{when } i+j \neq k \\ C_{AB}^C & \text{when } i+j = k \end{cases}$$

where  $i, j, k = 0, \dots, N$  correspond to the order of the expansion, and  $N$  is the truncation order. These structure constants can also be obtained within the  $S$ -expansion procedure. In order to achieve this, one must consider the  $0_S$ -reduction of an  $S$ -expanded algebra where  $S$  corresponds to the semigroup. The Maurer-Cartan forms power series expansion of an algebra  $\mathfrak{g}$ , with truncation order  $N$ , coincides with the  $0_S$ -reduction of the  $S_E^{(N)}$ -expanded algebra (see Ref. [9]). This is of course no coincidence. The set of powers of the rescaling parameter  $\lambda$ , together with the truncation at order  $N$ , satisfy precisely the multiplication law of  $S_E^{(N)}$ . As a matter of fact, we have  $\lambda^\alpha \lambda^\beta = \lambda^{\alpha+\beta}$  and the truncation can be imposed as  $\lambda^\alpha = 0$  when  $\alpha > N$ . It is for this reason that one must demand  $0_S T_A = 0$  in order to obtain the Maurer-Cartan expansion as an  $S_E^{(N)}$ -expansion: in this case the zero of the semigroup is the zero of the field as well.

The  $S$ -expansion procedure is valid no matter what the structure of the original  $\mathfrak{g}$  Lie algebra is, and in this sense it is very general. However, when something about the structure of  $\mathfrak{g}$  is known, a lot more can be done. As an example, in the context of Maurer-Cartan expansion, the rescaling and truncation can be performed in several ways depending on the structure of  $\mathfrak{g}$ , leading to several kinds of expanded algebras. Important examples of this are the generalized Inönü–Wigner contraction, or the  $M$  algebra as an expansion of  $osp(32|1)$  (see Refs. [2], [3]). This is also the case in the context of  $S$ -expansions. When some information about the structure of  $\mathfrak{g}$  is available, it is possible to find subalgebras of  $\mathfrak{G} = S_E^{(N)} \times \mathfrak{g}$  and other kinds of reduced algebras. In this way, all the algebras obtained by the Maurer-Cartan expansion procedure can be reobtained. New kinds of  $S$ -expanded algebras can also be obtained by considering semigroups different from  $S_E^{(N)}$ .

The purpose of this paper is to introduce a modification to the  $S$ -expansion method. The modification is carried out by imposing a condition on the  $S$ -expansion procedure, when the semigroup is given by a cyclic group of even order. The  $S$ -expanded algebras are called  $S_H$ -expanded algebras where  $S = Z_{2n}$ . The invariant tensors for  $S_H$ -expanded algebras are calculated and the dual formulation of  $S_H$ -expansion procedure is proposed.

Following Ref.[12], we consider the  $S_H$ -expansion of the five-dimensional  $AdS$  algebra and its corresponding invariants tensors are obtained. We also consider the study of a Chern-Simons Lagrangian invariant under the five-dimensional  $AdS$  algebra  $S_H$ -expanded and its relationship to the general relativity.

This paper is organized as follows: In Section 2 we review some aspects of the

Lie algebra  $S$ -expansion procedure. In Section 3 it is introduced a modification to the  $S$ -expansion method. In Section 4 the invariant tensors for  $S_H$ -expanded algebras are obtained and the dual formulation of  $S_H$ -expansion procedure is constructed. In Section 5 we consider the  $S_H$ -expansion of the five-dimensional  $AdS$  algebra and its corresponding invariants tensors are found. In Section 6 a Chern-Simons Lagrangian invariant under the five-dimensional  $AdS$  algebra  $S_H$ -expanded is constructed and its relationship to the general relativity is studied. A comment about possible developments and an appendix concludes the work.

## 2 The $S$ -expansion procedure

In this section we shall review the main aspects of the  $S$ -expansion procedure and their properties introduced in Ref. [9].

Let  $S = \{\lambda_\alpha\}$  be an abelian semigroup with 2-selector  $K_{\alpha\beta}{}^\gamma$  defined by

$$K_{\alpha\beta}{}^\gamma = \begin{cases} 1 & \lambda_\alpha \lambda_\beta = \lambda_\gamma \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

and  $\mathfrak{g}$  a Lie (super)algebra with basis  $\{T_A\}$  and structure constant  $C_{AB}{}^C$ ,

$$[T_A, T_B] = C_{AB}{}^C T_C. \quad (2)$$

Then it may be shown that the product  $\mathfrak{G} = S \times \mathfrak{g}$  is also a Lie (super)algebra with structure constants  $C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = K_{\alpha\beta}{}^\gamma C_{AB}{}^C$ ,

$$[T_{(A,\alpha)}, T_{(B,\beta)}] = C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} T_{(C,\gamma)}. \quad (3)$$

The proof is direct and may be found in Ref. [9].

**Definition 1** *Let  $S$  be an abelian semigroup and  $\mathfrak{g}$  a Lie algebra. The Lie algebra  $\mathfrak{G}$  defined by  $\mathfrak{G} = S \times \mathfrak{g}$  is called  $S$ -Expanded algebra of  $\mathfrak{g}$ .*

When the semigroup has a zero element  $0_S \in S$ , it plays a somewhat peculiar role in the  $S$ -expanded algebra. The above considerations motivate the following definition:

**Definition 2** *Let  $S$  be an abelian semigroup with a zero element  $0_S \in S$ , and let  $\mathfrak{G} = S \times \mathfrak{g}$  be an  $S$ -expanded algebra. The algebra obtained by imposing the condition  $0_S \mathbf{T}_A = 0$  on  $\mathfrak{G}$  (or a subalgebra of it) is called  $0_S$ -reduced algebra of  $\mathfrak{G}$  (or of the subalgebra).*

An  $S$ -expanded algebra has a fairly simple structure. Interestingly, there are at least two ways of extracting smaller algebras from  $S \times \mathfrak{g}$ . The first one gives rise to a *resonant subalgebra*, while the second produces reduced algebras. In particular, a resonant subalgebra can be obtained as follow.

Let  $\mathfrak{g} = \bigoplus_{p \in I} V_p$  be a decomposition of  $\mathfrak{g}$  in subspaces  $V_p$ , where  $I$  is a set of indices. For each  $p, q \in I$  it is always possible to define  $i_{(p,q)} \subset I$  such that

$$[V_p, V_q] \subset \bigoplus_{r \in i_{(p,q)}} V_r. \quad (4)$$

Now, let  $S = \bigcup_{p \in I} S_p$  be a subset decomposition of the abelian semigroup  $S$  such that

$$S_p \cdot S_q \subset \bigcup_{r \in i_{(p,q)}} S_p. \quad (5)$$

When such subset decomposition  $S = \bigcup_{p \in I} S_p$  exists, then we say that this decomposition is in resonance with the subspace decomposition of  $\mathfrak{g}$ ,  $\mathfrak{g} = \bigoplus_{p \in I} V_p$ .

The resonant subset decomposition is crucial in order to systematically extract subalgebras from the  $S$ -expanded algebra  $\mathfrak{G} = S \times \mathfrak{g}$ , as is proven in the following

**Theorem IV.2 of Ref. [9]:** Let  $\mathfrak{g} = \bigoplus_{p \in I} V_p$  be a subspace decomposition of  $\mathfrak{g}$ , with a structure described by eq. (4), and let  $S = \bigcup_{p \in I} S_p$  be a resonant subset decomposition of the abelian semigroup  $S$ , with the structure given in eq. (5). Define the subspaces of  $\mathfrak{G} = S \times \mathfrak{g}$ ,

$$W_p = S_p \times V_p, \quad p \in I. \quad (6)$$

Then,

$$\mathfrak{G}_R = \bigoplus_{p \in I} W_p \quad (7)$$

is a subalgebra of  $\mathfrak{G} = S \times \mathfrak{g}$ .

**Proof:** the proof may be found in Ref. [9].

**Definition 3** The algebra  $G_R = \bigoplus_{p \in I} W_p$  obtained is called a *Resonant Subalgebra* of the  $S$ -expanded algebra  $\mathfrak{G} = S \times \mathfrak{g}$ .

A useful property of the  $S$ -expansion procedure is that it provides us with an invariant tensor for the  $S$ -expanded algebra  $\mathfrak{G} = S \times \mathfrak{g}$  in terms of an invariant tensor for  $\mathfrak{g}$ . As shown in Ref. [9] the theorem VII.2 provide a general expression for an invariant tensor for a  $0_S$ -reduced algebra.

**Theorem VII.2 of Ref. [9]:** Let  $S$  be an abelian semigroup with nonzero elements  $\lambda_i$ ,  $i = 0, \dots, N$  and  $\lambda_{N+1} = 0_S$ . Let  $\mathfrak{g}$  be a Lie (super)algebra of basis  $\{T_A\}$ , and let  $\langle T_{A_1} \cdots T_{A_n} \rangle$  be an invariant tensor for  $\mathfrak{g}$ . The expression

$$\langle T_{(A_1, i_1)} \cdots T_{(A_n, i_n)} \rangle = \alpha_j K_{i_a \cdots i_n}^j \langle T_{A_1} \cdots T_{A_n} \rangle \quad (8)$$

where  $\alpha_j$  are arbitrary constants, corresponds to an invariant tensor for the  $0_S$ -reduced algebra obtained from  $\mathfrak{G} = S \times \mathfrak{g}$ .

**Proof:** the proof may be found in section 4.5 of Ref. [9].

### 3 The $S_H$ -expansion Method

In Ref. [9] it was found that given a Lie algebra  $\mathfrak{g}$  and a semigroup  $S$ , we can find a new Lie algebra  $S \times \mathfrak{g}$  called  $S$ -expanded Lie algebra. If the semigroup  $S$  is endowed with a zero element  $0_S$ , then it is possible to obtain a new Lie algebra by imposing the condition  $0_S \otimes T_A = 0$  on the  $S$ -expanded algebra.

In this section we consider the expansion of a Lie algebra  $\mathfrak{g}$  in the case that the semigroup  $S$  is given by a cyclic group with an even number of elements  $Z_{2n} = \{\lambda_0, \dots, \lambda_{2n-1}\}$ . We will prove that if we impose the condition  $\lambda_i \otimes T_A + \lambda_{i+n} \otimes T_A = 0$  on  $Z_{2n} \times \mathfrak{g}$ , then the resulting structure is also a Lie algebra. The condition will be called  $H$ -condition and the new Lie algebra obtained will be denoted by  $(Z_{2n} \times \mathfrak{g})_H$ .

The motivation for imposing the  $H$ -condition is based on the fact that, when the mechanism of  $S$ -expansion [9] is used, then it can be shown that the  $Z_2$ -expanded  $\mathfrak{so}(3)$ -algebra corresponds to an algebra isomorphic to the  $\mathfrak{so}(4)$  algebra

$$\mathfrak{so}(4) \simeq Z_2 \times \mathfrak{so}(3). \quad (9)$$

An open question is: is it possible to obtain, by  $S$ -expansion, the  $\mathfrak{so}(3, 1)$  algebra from the  $\mathfrak{so}(3)$  algebra?

To try to answer this question, consider the  $\mathfrak{so}(3, 1)$  algebra written in the form

$$[J_i, J_j] = \varepsilon_{ij}^k J_k, \quad [J_i, K_j] = \varepsilon_{ij}^k K_k, \quad [K_i, K_j] = -\varepsilon_{ij}^k J_k. \quad (10)$$

The fundamental difference that makes the  $\mathfrak{so}(4)$  and  $\mathfrak{so}(3, 1)$  algebras are not isomorphic is the minus sign appears on the right side of the commutator between the generators of the Lorentz boosts  $K_i$ .

Consider the following set of generators  $\{J_i, K_i, A_i, B_i\}$  where  $A_i = -J_i$  and  $B_i = -K_i$  and find the commutation relations between them:

$$\begin{aligned} [J_i, J_j] &= \varepsilon_{ij}^k J_k, & [J_i, K_j] &= \varepsilon_{ij}^k K_k, \\ [J_i, A_j] &= \varepsilon_{ij}^k A_k, & [J_i, B_j] &= \varepsilon_{ij}^k B_k, \\ [K_i, K_j] &= \varepsilon_{ij}^k A_k, & [K_i, A_j] &= \varepsilon_{ij}^k B_k, \\ [K_i, B_j] &= \varepsilon_{ij}^k J_k, & [A_i, A_j] &= \varepsilon_{ij}^k J_k, \\ [A_i, B_j] &= \varepsilon_{ij}^k K_k, & [B_i, B_j] &= \varepsilon_{ij}^k A_k. \end{aligned} \quad (11)$$

These commutation relations coincide with the commutation relations of the algebra  $Z_4 \times \mathfrak{so}(3)$ . In fact, using the  $S$ -expansion method with  $S = Z_4 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$  and  $J_{(i,\alpha)} = \lambda_\alpha J_i$  we have

$$\begin{aligned} [J_{(i,0)}, J_{(j,0)}] &= \varepsilon_{ij}^k J_{(k,0)}, & [J_{(i,0)}, J_{(j,1)}] &= \varepsilon_{ij}^k J_{(k,1)}, \\ [J_{(i,0)}, J_{(j,2)}] &= \varepsilon_{ij}^k J_{(k,2)}, & [J_{(i,0)}, J_{(j,3)}] &= \varepsilon_{ij}^k J_{(k,3)}, \\ [J_{(i,1)}, J_{(j,1)}] &= \varepsilon_{ij}^k J_{(k,2)}, & [J_{(i,1)}, J_{(j,2)}] &= \varepsilon_{ij}^k J_{(k,3)}, \\ [J_{(i,1)}, J_{(j,3)}] &= \varepsilon_{ij}^k J_{(k,0)}, & [J_{(i,2)}, J_{(j,2)}] &= \varepsilon_{ij}^k J_{(k,0)}, \\ [J_{(i,2)}, J_{(j,3)}] &= \varepsilon_{ij}^k J_{(k,1)}, & [J_{(i,3)}, J_{(j,3)}] &= \varepsilon_{ij}^k J_{(k,2)}, \end{aligned} \quad (12)$$

the commutation relations (11) and (12) coincide under the correspondence  $J_{(i,0)} = J_i$ ,  $J_{(i,1)} = K_i$ ,  $J_{(i,2)} = A_i$ ,  $J_{(i,3)} = B_i$ . The above result shows that if

we apply the conditions

$$J_{(i,2)} = -J_{(i,0)}, \quad J_{(i,3)} = -J_{(i,1)} \quad (13)$$

on the expanded algebra  $Z_4 \times \mathfrak{so}(3)$  we obtain the algebra  $\mathfrak{so}(3,1)$ .

It is interesting that these conditions are not independent. The second condition can be obtained if we multiply the first by  $\lambda_1$ , according to the law of multiplication of  $Z_4$ . Operating with any element  $Z_4$  get one any of the two conditions given in (13).

An interesting question is: do we get a Lie algebra if we apply this condition on arbitrary Lie algebra  $Z_4 \times \mathfrak{g}$ ?. To answer, we consider the  $Z_4$ -expansion of a Lie algebra  $\mathfrak{g}$ . Using the S-expansion method with  $S = Z_4 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$  and  $T_{(A,\alpha)} = \lambda_\alpha T_A$  we have

$$\begin{aligned} [T_{(A,0)}, T_{(B,0)}] &= C_{AB}^C T_{(C,0)}, & [T_{(A,0)}, T_{(B,1)}] &= C_{AB}^C T_{(C,1)}, \\ [T_{(A,0)}, T_{(B,2)}] &= C_{AB}^C T_{(C,2)}, & [T_{(A,0)}, T_{(B,3)}] &= C_{AB}^C T_{(C,3)}, \\ [T_{(A,1)}, T_{(B,1)}] &= C_{AB}^C T_{(C,2)}, & [T_{(A,1)}, T_{(B,2)}] &= C_{AB}^C T_{(C,3)}, \\ [T_{(A,1)}, T_{(B,3)}] &= C_{AB}^C T_{(C,0)}, & [T_{(A,2)}, T_{(B,2)}] &= C_{AB}^C T_{(C,0)}, \\ [T_{(A,2)}, T_{(B,3)}] &= C_{AB}^C T_{(C,1)}, & [T_{(A,3)}, T_{(B,3)}] &= C_{AB}^C T_{(C,2)}. \end{aligned} \quad (14)$$

Applying the conditions

$$T_{(A,2)} = -T_{(A,0)}, \quad T_{(A,3)} = -T_{(A,1)} \quad (15)$$

over (14), the following commutation relations are obtained

$$\begin{aligned} [T_{(A,0)}, T_{(B,0)}] &= C_{AB}^C T_{(C,0)}, & [T_{(A,0)}, T_{(B,1)}] &= C_{AB}^C T_{(C,1)}, \\ [T_{(A,0)}, (-T_{(B,0)})] &= C_{AB}^C (-T_{(C,0)}), & [T_{(A,0)}, (-T_{(B,1)})] &= C_{AB}^C (-T_{(C,1)}), \\ [T_{(A,1)}, T_{(B,1)}] &= C_{AB}^C (-T_{(C,0)}), & [T_{(A,1)}, (-T_{(B,0)})] &= (-T_{(C,1)}), \\ [T_{(A,1)}, (-T_{(B,1)})] &= C_{AB}^C T_{(C,0)}, & [(-T_{(A,0)}), (-T_{(B,0)})] &= C_{AB}^C T_{(C,0)}, \\ [(-T_{(A,0)}), (-T_{(B,1)})] &= C_{AB}^C T_{(C,1)}, & [(-T_{(A,1)}), (-T_{(B,1)})] &= C_{AB}^C (-T_{(C,0)}), \end{aligned} \quad (16)$$

which can be rewritten as

$$\begin{aligned} [T_{(A,0)}, T_{(B,0)}] &= C_{AB}^C T_{(C,0)}, & [T_{(A,0)}, T_{(B,1)}] &= C_{AB}^C T_{(C,1)}, \\ [T_{(A,1)}, T_{(B,1)}] &= -C_{AB}^C T_{(C,0)}. \end{aligned} \quad (17)$$

To verify that this algebra corresponds to a Lie algebra, we define  $T_\mu = (T_{(A,0)}, T_{(A,1)})$ , with  $\mu = 1, \dots, 2m$  and  $m = \dim \mathfrak{g}$ , whereupon the commutation relations (17) become  $[T_\mu, T_\nu] = f_{\mu\nu}^\rho T_\rho$ , with

$$\begin{aligned} f_{AB}^C &= C_{AB}^C, \\ f_{AB}^{C+m} &= 0, \\ f_{A(B+m)}^C &= f_{(A+m)B}^C = 0, \\ f_{A(B+m)}^{C+m} &= f_{(A+m)B}^{C+m} = C_{AB}^C, \\ f_{(A+m)(B+m)}^C &= -C_{AB}^C, \\ f_{(A+m)(B+m)}^{C+m} &= 0. \end{aligned} \quad (18)$$

From (18) one can see that the structure constants are antisymmetric in their low indices. It is straightforward to verify that they satisfy the Jacobi identity. In effect,

$$\begin{aligned}
& f_{\mu\nu}{}^\lambda f_{\lambda\rho}{}^\varepsilon + f_{\nu\rho}{}^\lambda f_{\lambda\mu}{}^\varepsilon + f_{\rho\mu}{}^\lambda f_{\lambda\nu}{}^\varepsilon \\
&= f_{\mu\nu}{}^A f_{A\rho}{}^\varepsilon + f_{\nu\rho}{}^A f_{A\mu}{}^\varepsilon + f_{\rho\mu}{}^A f_{A\nu}{}^\varepsilon + f_{\mu\nu}{}^{A+m} f_{(A+m)\rho}{}^\varepsilon \\
&+ f_{\nu\rho}{}^{A+m} f_{(A+m)\mu}{}^\varepsilon + f_{\rho\mu}{}^{A+m} f_{(A+m)\nu}{}^\varepsilon.
\end{aligned} \tag{19}$$

1. Case  $\varepsilon = E$ :

(a) if  $\mu = A, \nu = B, \rho = D$ , we have

$$C_{AB}{}^C C_{CD}{}^E + C_{BD}{}^C C_{CA}{}^E + C_{DA}{}^C C_{CB}{}^E = 0, \tag{20}$$

(b) if  $\mu = A, \nu = B, \rho = D + m$ , we have

$$0 = 0,$$

(c) if  $\mu = A, \nu = B + m, \rho = D + m$ , we have

$$C_{BD}{}^C C_{CA}{}^E + C_{AB}{}^C C_{CD}{}^E + C_{DA}{}^C C_{CB}{}^E = 0, \tag{21}$$

(d) if  $\mu = A + m, \nu = B + m, \rho = D + m$ , we have

$$0 = 0,$$

2. Case  $\varepsilon = E + m$ :

(a) if  $\mu = A, \nu = B, \rho = D$ , we have

$$0 = 0.$$

(b) if  $\mu = A, \nu = B, \rho = D + m$ , we have

$$C_{AB}{}^C C_{CD}{}^E + C_{BD}{}^C C_{CA}{}^E + C_{DA}{}^C C_{CB}{}^E = 0, \tag{22}$$

(c) if  $\mu = A, \nu = B + m, \rho = D + m$ , we have

$$0 = 0,$$

(d) if  $\mu = A + m, \nu = B + m, \rho = D + m$ , we have

$$C_{AB}{}^C C_{CD}{}^E + C_{BD}{}^C C_{CA}{}^E + C_{DA}{}^C C_{CB}{}^E = 0. \tag{23}$$

This proves that, when the condition (15) is imposed on an arbitrary Lie algebra  $Z_4 \times \mathfrak{g}$ , a new Lie algebra, which has half of the generators of the  $Z_4 \times \mathfrak{g}$  algebra, is obtained.

## 4 $H$ -Condition on $Z_{2n}$ -expanded algebras

Now consider the generalization to the case of an arbitrary even-order cyclic group. We have seen that the imposition of the  $H$  condition (15) on the expanded algebra generates a new algebra. The generalization to the case of an arbitrary cyclic group of even order can be carried out by rewriting  $Z_{2n}$ , in the form

$$Z_{2n} = \{\lambda_0, \dots, \lambda_{n-1}\} \cup \{\lambda_n, \dots, \lambda_{2n-1}\}, \quad n \in N \quad (24)$$

$$Z_{2n} = \{\lambda_i\} \cup \{\lambda_{i+n}\}, \quad 0 \leq i \leq n-1 \quad (25)$$

and then express the generators generated by the elements  $\{\lambda_n, \dots, \lambda_{2n-1}\}$ , which we will call "*greater interval*", in terms of generators generated by elements  $\{\lambda_0, \dots, \lambda_{n-1}\}$ , which we will call "*minor interval*".

The generalization of  $H$  condition(15) is given by

$$T_{(A, i+n)} = -T_{(A, i)}. \quad (26)$$

Since the law of multiplication of  $Z_{2n}$  is given by

$$\lambda_\alpha \lambda_\beta = \lambda_{\gamma \equiv \alpha + \beta \pmod{2n}} \quad (27)$$

we have that if we multiply equation (26) for arbitrary element  $\lambda_\alpha \in Z_{2n}$ , we have (remember that  $T_{(A, \alpha)} = \lambda_\alpha \times T_A$ ):

1. Case  $\alpha = j$

$$\begin{aligned} T_{(A, j+n+i \pmod{2n})} &= -T_{(A, j+i \pmod{2n})}, \\ T_{(A, (j+i)+n \pmod{2n})} &= -T_{(A, j+i \pmod{2n})}. \end{aligned} \quad (28)$$

Because  $\lambda_{j+i \pmod{2n}} \in Z_{2n}$ , we always obtain a non-trivial condition.

2. Case  $\alpha = j + n$

$$\begin{aligned} T_{(A, j+n+i+n \pmod{2n})} &= -T_{(A, j+n+i \pmod{2n})}, \\ T_{(A, j+i \pmod{2n})} &= -T_{(A, (j+i)+n \pmod{2n})}, \end{aligned} \quad (29)$$

so that

$$T_{(A, (j+i)+n \pmod{2n})} = -T_{(A, j+i \pmod{2n})}. \quad (30)$$

From the closure property we have  $\lambda_{j+i \pmod{2n}} \in Z_{2n}$ , so in this case we also obtain a non-trivial condition.

The consistency condition (26) together with the previous discussions allow us to establish the following theorem:

**Theorem 1:** Let  $\mathfrak{g} = \{T_A\}$  be a Lie algebra, with  $A = 1, \dots, \dim \mathfrak{g}$  and let  $Z_{2n} = \{\lambda_\alpha\}$  be a cyclic un group with  $n \in N$  and  $\alpha = 0, \dots, 2n-1$ . The Algebra  $\{T_{(A, i)}\}$ ,  $i = 0, \dots, n-1$ , resulting from imposing the condition (26)



on the algebra expanded  $Z_{2n} \times \mathfrak{g}$  is a Lie algebra whose generators satisfy the following commutation relations

$$\begin{aligned} [T_{(A,i)}, T_{(B,j)}] &= (K_{ij}^k - K_{ij}^{k+n}) C_{AB}^C T_{(C,k)}, \\ A, B, C &= 1, \dots, \dim \mathfrak{g}, \quad i, j, k = 0, \dots, n-1. \end{aligned} \quad (31)$$

**Proof:** The components  $i, j = 0, \dots, n-1$  of the commutations relations for the expanded algebra are given by

$$\begin{aligned} [T_{(A,i)}, T_{(B,j)}] &= K_{ij}^\gamma C_{AB}^C T_{(C,\gamma)}, \\ &= K_{ij}^k C_{AB}^C T_{(C,k)} + K_{ij}^{k+n} C_{AB}^C T_{(C,k+n)} \\ &= K_{ij}^k C_{AB}^C T_{(C,k)} - K_{ij}^{k+n} C_{AB}^C T_{(C,k)} \\ &= (K_{ij}^k - K_{ij}^{k+n}) C_{AB}^C T_{(C,k)}, \end{aligned} \quad (32)$$

where we have used the condition (26). The next step is to prove that the structure constants

$$C_{(A,i)(B,j)}^{(C,k)} = (K_{ij}^k - K_{ij}^{k+n}) C_{AB}^C \quad (33)$$

satisfy the antisymmetry property in its lower indices and the Jacobi identity. From equation (33) we have

**Proof.**

$$\begin{aligned} C_{(A,i)(B,j)}^{(C,k)} &= (K_{ij}^k - K_{ij}^{k+n}) C_{AB}^C \\ &= (K_{ji}^k - K_{ji}^{k+n}) C_{AB}^C \\ &= -(K_{ji}^k - K_{ji}^{k+n}) C_{BA}^C \\ &= -C_{(B,j)(A,i)}^{(C,k)}, \end{aligned} \quad (34)$$

which proves that the structure constants of the algebra  $(Z_{2n} \times \mathfrak{g})_H = \{T_{(A,i)}\}_{i=1}^{n-1}$  are antisymmetric in their lower indices. To test the Jacobi identity consider the following expression

$$\begin{aligned} &C_{(A,i)(B,j)}^{(C,k)} C_{(C,k)(D,l)}^{(E,m)} + C_{(B,j)(D,l)}^{(C,k)} C_{(C,k)(A,i)}^{(E,m)} \\ &+ C_{(D,l)(A,i)}^{(C,k)} C_{(C,k)(B,j)}^{(E,m)} \\ &= (K_{ij}^k - K_{ij}^{k+n}) C_{AB}^C (K_{kl}^m - K_{kl}^{m+n}) C_{CD}^E \\ &+ (K_{jl}^k - K_{jl}^{k+n}) C_{BD}^C (K_{ki}^m - K_{ki}^{m+n}) C_{CA}^E \\ &+ (K_{li}^k - K_{li}^{k+n}) C_{DA}^C (K_{kj}^m - K_{kj}^{m+n}) C_{CB}^E \end{aligned} \quad (35)$$

from where

$$\begin{aligned}
& C_{(A,i)(B,j)}^{(C,k)} C_{(C,k)(D,l)}^{(E,m)} + C_{(B,j)(D,l)}^{(C,k)} C_{(C,k)(A,i)}^{(E,m)} \\
& + C_{(D,l)(A,i)}^{(C,k)} C_{(C,k)(B,j)}^{(E,m)} \\
& = (K_{ij}^k K_{kl}^m - K_{ij}^k K_{kl}^{m+n} - K_{ij}^{k+n} K_{kl}^m + K_{ij}^{k+n} K_{kl}^{m+n}) C_{AB}^C C_{CD}^E \\
& + (K_{jl}^k K_{ki}^m - K_{jl}^k K_{ki}^{m+n} - K_{jl}^{k+n} K_{ki}^m + K_{jl}^{k+n} K_{ki}^{m+n}) C_{BD}^C C_{CA}^E \\
& + (K_{li}^k K_{kj}^m - K_{li}^k K_{kj}^{m+n} - K_{li}^{k+n} K_{kj}^m + K_{li}^{k+n} K_{kj}^{m+n}) C_{DA}^C C_{CB}^E.
\end{aligned} \tag{36}$$

Since the selectors satisfy the relation (see [9])  $K_{\alpha\beta\gamma}^\delta = K_{\alpha\beta}^\varepsilon K_{\varepsilon\gamma}^\delta$  we have

$$\begin{aligned}
K_{ijl}^m &= K_{ij}^k K_{kl}^m + K_{ij}^{k+n} K_{k+n,l}^m \\
K_{ijl}^{m+n} &= K_{ij}^k K_{kl}^{m+n} + K_{ij}^{k+n} K_{k+n,l}^{m+n}
\end{aligned} \tag{37}$$

so that

$$\begin{aligned}
& C_{(A,i)(B,j)}^{(C,k)} C_{(C,k)(D,l)}^{(E,m)} + C_{(B,j)(D,l)}^{(C,k)} C_{(C,k)(A,i)}^{(E,m)} \\
& + C_{(D,l)(A,i)}^{(C,k)} C_{(C,k)(B,j)}^{(E,m)} \\
& = (K_{ijl}^m - K_{ij}^{k+n} K_{k+n,l}^m - K_{ijl}^{m+n} + K_{ij}^{k+n} K_{k+n,l}^{m+n} \\
& - K_{ij}^{k+n} K_{kl}^m + K_{ij}^{k+n} K_{kl}^{m+n}) C_{AB}^C C_{CD}^E \\
& + (K_{jli}^m - K_{jl}^{k+n} K_{k+n,i}^m - K_{jli}^{m+n} + K_{jl}^{k+n} K_{k+n,i}^{m+n} \\
& - K_{jl}^{k+n} K_{ki}^m + K_{jl}^{k+n} K_{ki}^{m+n}) C_{BD}^C C_{CA}^E \\
& + (K_{lij}^m - K_{li}^{k+n} K_{k+n,j}^m - K_{lij}^{m+n} + K_{li}^{k+n} K_{k+n,j}^{m+n} \\
& - K_{li}^{k+n} K_{kj}^m + K_{li}^{k+n} K_{kj}^{m+n}) C_{DA}^C C_{CB}^E.
\end{aligned} \tag{38}$$

Taking into account that  $K_{\alpha\beta\gamma}^\delta = K_{\beta\gamma\alpha}^\delta = K_{\gamma\alpha\beta}^\delta$  (see [9]) and that the algebra  $\mathfrak{g}$  satisfies the identity Jacobi, we have

$$\begin{aligned}
& C_{(A,i)(B,j)}^{(C,k)} C_{(C,k)(D,l)}^{(E,m)} + C_{(B,j)(D,l)}^{(C,k)} C_{(C,k)(A,i)}^{(E,m)} \\
& + C_{(D,l)(A,i)}^{(C,k)} C_{(C,k)(B,j)}^{(E,m)} \\
& = K_{ij}^{k+n} (-K_{k+n,l}^m + K_{k+n,l}^{m+n} - K_{kl}^m + K_{kl}^{m+n}) C_{AB}^C C_{CD}^E \\
& + K_{jl}^{k+n} (-K_{k+n,i}^m + K_{k+n,i}^{m+n} - K_{ki}^m + K_{ki}^{m+n}) C_{BD}^C C_{CA}^E \\
& + (-K_{k+n,j}^m + K_{k+n,j}^{m+n} - K_{kj}^m + K_{kj}^{m+n}) C_{DA}^C C_{CB}^E
\end{aligned} \tag{39}$$

Since  $K_{k+n,l}^m = K_{kl}^{m+n}$  and  $K_{k+n,l}^{m+n} = K_{kl}^m$  (see appendix) we have each parenthesis of the above equation vanishes, proving the Jacobi identity. ■

#### 4.1 Lorentz algebra from $so(3)$ algebra

Using the S-expansion procedure with  $S = Z_4 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$  is found that the generators  $J_{(i,\alpha)} = \lambda_\alpha J_i$  of the  $Z_4 \times \mathfrak{so}(3)$  algebra satisfy the following

commutation relations

$$\begin{aligned}
\begin{aligned}
[J_{(i,0)}, J_{(j,0)}] &= \varepsilon_{ij}^k J_{(k,0)}, \\
[J_{(i,0)}, J_{(j,2)}] &= \varepsilon_{ij}^k J_{(k,2)}, \\
[J_{(i,1)}, J_{(j,1)}] &= \varepsilon_{ij}^k J_{(k,2)}, \\
[J_{(i,1)}, J_{(j,3)}] &= \varepsilon_{ij}^k J_{(k,0)}, \\
[J_{(i,2)}, J_{(j,3)}] &= \varepsilon_{ij}^k J_{(k,1)},
\end{aligned}
\quad
\begin{aligned}
[J_{(i,0)}, J_{(j,1)}] &= \varepsilon_{ij}^k J_{(k,1)}, \\
[J_{(i,0)}, J_{(j,3)}] &= \varepsilon_{ij}^k J_{(k,3)}, \\
[J_{(i,1)}, J_{(j,2)}] &= \varepsilon_{ij}^k J_{(k,3)}, \\
[J_{(i,2)}, J_{(j,2)}] &= \varepsilon_{ij}^k J_{(k,0)}, \\
[J_{(i,3)}, J_{(j,3)}] &= \varepsilon_{ij}^k J_{(k,2)},
\end{aligned}
\end{aligned} \tag{40}$$

Imposing the condition

$$J_{(i,2)} = -J_{(i,0)}, \quad J_{(i,3)} = -J_{(i,1)} \tag{41}$$

on the expanded algebra  $Z_4 \times \mathfrak{so}(3)$  we obtain

$$\mathfrak{so}(3, 1) \simeq (Z_4 \times \mathfrak{so}(3))_H. \tag{42}$$

## 4.2 Dimension of a $S_H$ -expanded algebra

From [9] we know that if  $\mathfrak{g}$  is a Lie algebra of dimension  $\dim \mathfrak{g}$ , then the dimension of  $S$ -expanded algebra  $S \times \mathfrak{g}$  is given by  $|S| \dim \mathfrak{g}$ . In the case that  $S = Z_{2n}$  we have

$$\dim(Z_{2n} \times \mathfrak{g})_H = n \dim \mathfrak{g}. \tag{43}$$

In fact, since the dimension of the algebra  $Z_{2n} \times \mathfrak{g}$  is  $2n \dim \mathfrak{g}$ , we have that if we impose the  $H$  condition on  $Z_{2n} \times \mathfrak{g}$ , then it follows that the number of generators  $T_{(A,i)}$  is halved, i.e.,  $i = 0, 1, \dots, n-1$ . Therefore the dimension of the  $S_H$ -expanded algebra is given by  $n \dim \mathfrak{g}$ .

## 4.3 Construction with the greater interval

The construction of an algebra  $S_H$ -expanded is performed with generators obtained using only elements minor interval. It is of interest to analyze the characteristics that has a algebra constructed from generators obtained using only elements greater interval.

Evaluating the components  $i + n$  of the commutation relations for  $Z_{2n}$ -expanded algebra we have

$$\begin{aligned}
[T_{(A,i+n)}, T_{(B,j+n)}] &= K_{i+n,j+n}^\gamma C_{AB}^C T_{(C,\gamma)}, \\
&= K_{i+n,j+n}^k C_{AB}^C T_{(C,k)} + K_{i+n,j+n}^{k+n} C_{AB}^C T_{(C,k+n)}, \\
&= -K_{i+n,j+n}^k C_{AB}^C T_{(C,k+n)} + K_{i+n,j+n}^{k+n} C_{AB}^C T_{(C,k+n)}, \\
&= -(K_{ij}^k - K_{ij}^{k+n}) C_{AB}^C T_{(C,k+n)}.
\end{aligned} \tag{44}$$

From (44) can see that when changing base  $T'_{(A,i+n)} = -T_{(A,i+n)}$  we obtain an algebra isomorphic to the algebra  $S_H$ -expanded, generated by generators  $T_{(A,i)}$ .

#### 4.4 $S_H$ -expansion in the case that $S = Z_2$

If  $n = 1$ , the  $S_H$ -expansion corresponds to the trivial case, i.e.,

$$(Z_2 \times \mathfrak{g})_H \simeq \mathfrak{g}. \quad (45)$$

In fact, from the previous theorem we can see that the commutation relations  $(Z_2 \times \mathfrak{g})_H$  are given by

$$\begin{aligned} [T_{(A,0)}, T_{(B,0)}] &= (K_{00}^k - K_{00}^{k+1}) C_{AB}^C T_{(C,k)}, \\ &= (K_{00}^0 - K_{00}^1) C_{AB}^C T_{(C,0)}, \\ &= C_{AB}^C T_{(C,0)}, \end{aligned} \quad (46)$$

which proves that the obtained algebra is an algebra isomorphic to  $\mathfrak{g}$ .

#### 4.5 The Klein group

The Klein group  $D_4$  corresponds to the direct product  $Z_2 \times Z_2$  (see Appendix 1). Taking into account that the  $S'$ -expansion, of a  $S$ -expanded algebra, corresponds to an  $(S' \times S)$ -expansion, we have

$$D_4 \times \mathfrak{g} \simeq Z_2 \times (Z_2 \times \mathfrak{g}). \quad (47)$$

From (45) we know that applying the  $H$  condition on an algebra  $Z_2$ -expanded, we get an algebra isomorphic to the original algebra. This means that

$$(D_4 \times \mathfrak{g})_H \simeq (Z_2 \times (Z_2 \times \mathfrak{g}))_H \simeq Z_2 \times \mathfrak{g}. \quad (48)$$

It is straightforward to prove that the imposition of the condition  $T_{(A,2)} = -T_{(A,0)}$  on  $D_4 \times \mathfrak{g}$ , leads to a Lie algebra  $Z_2$ -expanded. Since the  $S_H$ -expansion procedure has been so far defined only for cyclic groups, this result suggests the possibility of generalizing the  $S_H$ -expansion procedure to the case of a larger family of abelian groups.

Since a  $Z_2$ -expansion (which corresponds to a " $(D_4)_H$ -expansion") of  $\mathfrak{so}(3)$  algebra leads to algebra  $\mathfrak{so}(4)$  and that a  $(Z_4)_H$ -expansion of the  $\mathfrak{so}(3)$  algebra leads to the algebra  $\mathfrak{so}(3,1)$  we can write

$$\mathfrak{so}(3,1) \simeq (Z_4 \times \mathfrak{so}(3))_H, \quad \mathfrak{so}(4) \simeq (D_4 \times \mathfrak{so}(3))_H, \quad (49)$$

where  $Z_4$  and  $D_4$  are the only groups of order 4 that there, up to isomorphism.

### 5 Invariant tensors and dual formulation of $S_H$ -expansion procedure

In this section, we obtain the invariant tensors corresponding to  $S_H$ -expanded algebras. The dual formulation of  $S_H$ -expansion procedure is also considered.

## 5.1 Invariant Tensors

In ref. [9] was proved that the invariant tensors, corresponding to an  $S$ -expanded algebra, can be obtained from the invariant tensors of the original algebra. We now prove that this result remains valid in the case of  $S_H$ -expanded algebras.

**Theorem 2:** Let  $\langle T_{A_1}, \dots, T_{A_r} \rangle$  be an invariant tensor for a Lie algebra  $\mathfrak{g}$  of basis  $\{T_A\}$ . Then the expression

$$\langle T_{(A_1, i_1)}, \dots, T_{(A_r, i_r)} \rangle = \alpha_\gamma K_{i_1 \dots i_r}^\gamma \langle T_{A_1}, \dots, T_{A_r} \rangle, \quad (50)$$

where  $K_{\alpha_1 \dots \alpha_r}^\gamma$  is the  $r$ -selector for  $Z_{2n}$ , corresponds to an invariant tensor for the  $S_H$ -expanded algebra  $(Z_{2n} \times \mathfrak{g})_H$ , with  $i_1, \dots, i_r \in \{0, \dots, n-1\}$  and  $0 \leq i_k \leq n-1$ .

**Proof:** The invariance condition for  $\langle T_{A_1}, \dots, T_{A_r} \rangle$  under  $\mathfrak{g}$  reads

$$\sum_{p=1}^r X_{A_0 \dots A_r}^{(p)} = 0, \quad (51)$$

where

$$X_{A_0 \dots A_r}^{(p)} = C_{A_0 A_p}^B \langle T_{A_1} \dots T_{A_{p-1}} T_B T_{A_{p+1}} \dots T_{A_r} \rangle. \quad (52)$$

Define now

$$\begin{aligned} X_{(A_0, \alpha_0) \dots (A_r, \alpha_r)}^{(p)} &= C_{(A_0, i_0)(A_p, i_p)}^{(B, \beta)} \langle T_{(A_1, i_1)} \dots T_{(A_{p-1}, i_{p-1})} T_{(B, \beta)} T_{(A_{p+1}, i_{p+1})} \dots T_{(A_r, \alpha_r)} \rangle \\ &= C_{(A_0, i_0)(A_p, i_p)}^{(B, j)} \langle T_{(A_1, i_1)} \dots T_{(A_{p-1}, i_{p-1})} T_{(B, j)} T_{(A_{p+1}, i_{p+1})} \dots T_{(A_r, \alpha_r)} \rangle \\ &\quad + C_{(A_0, i_0)(A_p, i_p)}^{(B, j+n)} \langle T_{(A_1, i_1)} \dots T_{(A_{p-1}, i_{p-1})} T_{(B, j+n)} T_{(A_{p+1}, i_{p+1})} \dots T_{(A_r, \alpha_r)} \rangle. \end{aligned}$$

Imposing the  $H$ -condition, we have

$$\begin{aligned} X_{(A_0, \alpha_0) \dots (A_r, \alpha_r)}^{(p)} &= C_{(A_0, i_0)(A_p, i_p)}^{(B, j)} \langle T_{(A_1, i_1)} \dots T_{(A_{p-1}, i_{p-1})} T_{(B, j)} T_{(A_{p+1}, i_{p+1})} \dots T_{(A_r, \alpha_r)} \rangle \\ &\quad - C_{(A_0, i_0)(A_p, i_p)}^{(B, j+n)} \langle T_{(A_1, i_1)} \dots T_{(A_{p-1}, i_{p-1})} T_{(B, j)} T_{(A_{p+1}, i_{p+1})} \dots T_{(A_r, \alpha_r)} \rangle \end{aligned}$$

and replacing the expressions  $C_{(A, \alpha)(B, \beta)}^{(C, \gamma)} = K_{\alpha\beta}^\gamma C_{AB}^C$  for the  $S$ -expansion structure constants and (52) for  $\langle T_{(A_1, i_1)} \dots T_{(A_n, i_n)} \rangle$ , we get

$$\begin{aligned} X_{(A_0, i_0) \dots (A_n, i_n)}^{(p)} &= \left( K_{i_0 i_p}^j - K_{i_0 i_p}^{j+n} \right) C_{A_0 A_p}^B \alpha_\gamma K_{i_1 \dots i_{p-1} j i_{p+1} \dots i_r}^\gamma \langle T_{A_1} \dots T_{A_{p-1}} T_B T_{A_{p+1}} \dots T_{A_r} \rangle \\ X_{(A_0, i_0) \dots (A_n, i_n)}^{(p)} &= \alpha_\gamma \left( K_{i_0 i_p}^j - K_{i_0 i_p}^{j+n} \right) K_{i_1 \dots i_{p-1} j i_{p+1} \dots i_r}^\gamma X_{A_0 \dots A_r}^{(p)} \end{aligned}$$

and from (51) one readily concludes that

$$\sum_{p=1}^n X_{(A_0, \alpha_0) \dots (A_n, \alpha_n)}^{(p)} = 0. \quad (53)$$

Therefore,  $\langle T_{(A_1, i_1)} \dots T_{(A_n, i_n)} \rangle = \alpha_\gamma K_{i_1 \dots i_n}^\gamma \langle T_{A_1} \dots T_{A_n} \rangle$  is an invariant tensor for  $\mathfrak{G} = Z_{2n} \otimes \mathfrak{g}$ .

## 6 Dual formulation of the $S_H$ -expansion procedure

In Ref. [11] was found a dual formulation of the Lie algebra  $S$ -expansion procedure, which is based on the dual picture of a Lie algebra given by the Maurer-Cartan forms. In this section we consider the dual formulation of the  $S_H$ -expansion procedure.

**Theorem 3:** Let  $\omega^{(A,\gamma)}$  be the Maurer-Cartan forms for the expanded Lie algebra  $Z_{2n} \times \mathfrak{g}$ , which satisfy the Maurer-Cartan equations

$$d\omega^{(C,\gamma)} + \frac{1}{2}K_{\alpha\beta}{}^\gamma C_{AB}{}^C \omega^{(A,\alpha)} \omega^{(B,\beta)} = 0 \quad (54)$$

Imposing the  $H$ -condition

$$\omega^{(A,i+n)} \stackrel{!}{=} -\omega^{(A,i)}; \quad i = 1, \dots, n-1 \quad (55)$$

on the expanded Lie algebra  $Z_{2n} \times \mathfrak{g}$  we obtain the Maurer-Cartan equations for the  $S_H$ -expanded Lie algebra  $(Z_{2n} \times \mathfrak{g})_H$ .

**Proof:** From equation (54) we can see that the components  $k = 0, \dots, n-1$  of the Maurer-Cartan equations for the  $S$ -expanded Lie algebra are

$$d\omega^{(C,k)} + \frac{1}{2}K_{\alpha\beta}{}^k C_{AB}{}^C \omega^{(A,\alpha)} \omega^{(B,\beta)} = 0. \quad (56)$$

Since

$$\begin{aligned} K_{\alpha\beta}{}^k \omega^{(A,\alpha)} \omega^{(B,\beta)} &= K_{i\beta}{}^k \omega^{(A,i)} \omega^{(B,\beta)} + K_{i+n,\beta}{}^k \omega^{(A,i+n)} \omega^{(B,\beta)} \\ &= K_{ij}{}^k \omega^{(A,i)} \omega^{(B,j)} + K_{i,j+n}{}^k \omega^{(A,i)} \omega^{(B,j+n)} \\ &\quad + K_{i+n,j}{}^k \omega^{(A,i+n)} \omega^{(B,j)} + K_{i+n,j+n}{}^k \omega^{(A,i+n)} \omega^{(B,j+n)} \end{aligned}$$

Imposing the  $H$ -condition (55), we have

$$K_{\alpha\beta}{}^k \omega^{(A,\alpha)} \omega^{(B,\beta)} = (K_{ij}{}^k - K_{i,j+n}{}^k - K_{i+n,j}{}^k + K_{i+n,j+n}{}^k) \omega^{(A,i)} \omega^{(B,j)}.$$

So that Eq. (56) takes the form

$$d\omega^{(C,k)} + \frac{1}{2} (K_{ij}{}^k - K_{i,j+n}{}^k - K_{i+n,j}{}^k + K_{i+n,j+n}{}^k) C_{AB}{}^C \omega^{(A,i)} \omega^{(B,j)} = 0.$$

Since the selectors of  $S = Z_{2n}$  satisfy the equations  $K_{i+n,j}{}^k = K_{ij}{}^{k+n}$ ;  $K_{i+n,j+n}{}^\gamma = K_{ij}{}^\gamma$ ;  $K_{i,j+n}{}^k = K_{i+n,j}{}^k$  we have

$$d\omega^{(C,k)} + \frac{1}{2} [2 (K_{ij}{}^k - K_{ij}{}^{k+n})] C_{AB}{}^C \omega^{(A,i)} \omega^{(B,j)} = 0.$$

Therefore  $\omega^{(A,i)}$  are the Maurer-Cartan forms for a Lie algebra, which is isomorphic to  $(Z_{2n} \times \mathfrak{g})_H$  whose structure constants are given by

$$C_{(A,i)(B,i)}^{(C,\gamma)} = [2 (K_{ij}{}^k - K_{ij}{}^{k+n})] C_{AB}{}^C.$$

## 7 General Relativity with cosmological constant from Chern-Simons gravity

According to the principles of general relativity (GR), the spacetime is a dynamical object which has independent degrees of freedom, and is governed by dynamical equations, namely the Einstein field equations. This means that in GR the geometry is dynamically determined. Therefore, the construction of a gauge theory of gravity requires an action that does not consider a fixed spacetime background. A five dimensional action for gravity fulfilling these conditions is the five-dimensional Chern–Simons AdS gravity action, which can be written as

$$L_{\text{AdS}}^{(5)} = \kappa \left( \frac{1}{5l^5} \epsilon_{a_1 \dots a_5} e^{a_1} \dots e^{a_5} + \frac{2}{3l^3} \epsilon_{a_1 \dots a_5} R^{a_1 a_2} e^{a_3} \dots e^{a_5} + \frac{1}{l} \epsilon_{a_1 \dots a_5} R^{a_1 a_2} R^{a_3 a_4} e^{a_5} \right), \quad (57)$$

where  $e^a$  corresponds to the 1-form *vielbein*, and  $R^{ab} = d\omega^{ab} + \omega_c^a \omega^{cb}$  to the Riemann curvature in the first order formalism [13], [14], [15].

If Chern-Simons theories are the appropriate gauge-theories to provide a framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to General Relativity.

In ref. [12] was shown that the standard, five-dimensional General Relativity, *without a cosmological term*, can be obtained from Chern-Simons gravity theory for a certain Lie algebra  $\mathcal{B}$ . The Chern-Simons Lagrangian is built from a  $\mathcal{B}$ -valued, one-form gauge connection  $A$  which depends on a scale parameter  $l$  which can be interpreted as a coupling constant that characterizes different regimes within the theory. The  $\mathcal{B}$  algebra, on the other hand, is obtained from the *AdS* algebra and a particular semigroup  $S$  by means of the *S*-expansion procedure introduced in refs. [9], [11]. The field content induced by  $\mathcal{B}$  includes the vielbein  $e^a$ , the spin connection  $\omega^{ab}$  and two extra bosonic fields  $h^a$  and  $k^{ab}$ .

The five dimensional Chern-Simons Lagrangian for the  $\mathcal{B}$  algebra is given by [12]:

$$L_{\text{ChS}}^{(5)} = \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcde} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right), \quad (58)$$

where we can see that (i) if one identifies the field  $e^a$  with the vielbein, the system consists of the Einstein-Hilbert action, without a cosmological, constant plus nonminimally coupled matter fields given by  $h^a$  and  $k^{ab}$ ; (ii) it is possible to recover the odd-dimensional Einstein gravity theory from a Chern-Simons gravity theory in the limit where the coupling constant  $l$  equals to zero while keeping the effective Newton's constant fixed.

In this section it is shown that the five-dimensional Einstein-Hilbert action *with a cosmological term* can be obtained from a Chern-Simons gravity action, invariant under a Lie algebra obtained by  $S_H$ -expansion of the *AdS* algebra.

## 7.1 $S_H$ -expansion of $AdS_5$ algebra

From Ref.[12] we know that the  $\mathfrak{B}_5$ -algebra can be obtained by  $S_E^{(3)}$ -expansion resonant and reduced of the  $AdS$  algebra, i.e.,

$$\mathfrak{B}_5 = \left( S_E^{(3)} \times AdS_5 \right)_{R,0_S}, \quad (59)$$

where  $R,0_S$  denotes resonance followed by a  $0_S$ -reduction. This algebra has 30 generators which are denoted by  $J_{ab}$ ,  $Z_{ab}$ ,  $P_a$  and  $Z_a$  with  $a, b = 1, \dots, 5$ . Now we consider a new algebra, which has also 30 generators, and that can be obtained by  $S_H$ -expansion of the  $AdS_5$  algebra, which will be denoted by  $\mathfrak{C}_5$ , where

$$\mathfrak{C}_5 = (Z_4 \times AdS_5)_H. \quad (60)$$

The generators of  $AdS_5$  algebra satisfy the following commutation relation

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = f_{ab,cd}{}^{ef} \tilde{J}_{ef}, \quad [\tilde{J}_{ab}, \tilde{P}_c] = f_{ab,c}{}^d \tilde{P}_d, \quad [\tilde{P}_a, \tilde{P}_b] = \tilde{J}_{ab},$$

where

$$f_{ab,cd}{}^{ef} = -\frac{1}{2} \left\{ \eta_{ac} (\delta_b^e \delta_d^f - \delta_d^e \delta_b^f) + \eta_{bd} (\delta_a^e \delta_c^f - \delta_c^e \delta_a^f) + \eta_{cb} (\delta_d^e \delta_a^f - \delta_a^e \delta_d^f) + \eta_{da} (\delta_c^e \delta_b^f - \delta_b^e \delta_c^f) \right\}$$

$$f_{ab,c}{}^e = -(\eta_{ac} \delta_b^e - \eta_{bc} \delta_a^e)$$

The corresponding commutation relation of the generators of the  $S_H$ -expanded  $AdS_5$  algebra, which will be denoted by  $\mathfrak{C}_5$ , are given by

$$\begin{aligned} \left[ \tilde{J}_{(ab,0)}, \tilde{J}_{(cd,0)} \right] &= f_{ab,cd}{}^{ef} \tilde{J}_{(ef,0)}, & \left[ \tilde{J}_{(ab,1)}, \tilde{P}_{(c,0)} \right] &= f_{ab,c}{}^d \tilde{P}_{(d,1)}, \\ \left[ \tilde{J}_{(ab,0)}, \tilde{J}_{(cd,1)} \right] &= f_{ab,cd}{}^{ef} \tilde{J}_{(ef,1)}, & \left[ \tilde{J}_{(ab,1)}, \tilde{P}_{(c,1)} \right] &= -f_{ab,c}{}^d \tilde{P}_{(d,0)}, \\ \left[ \tilde{J}_{(ab,1)}, \tilde{J}_{(cd,1)} \right] &= -f_{ab,cd}{}^{ef} \tilde{J}_{(ef,0)}, & \left[ \tilde{P}_{(a,0)}, \tilde{P}_{(b,0)} \right] &= \tilde{J}_{(ab,0)}, \\ \left[ \tilde{J}_{(ab,0)}, \tilde{P}_{(c,0)} \right] &= f_{ab,c}{}^d \tilde{P}_{(d,0)}, & \left[ \tilde{P}_{(a,0)}, \tilde{P}_{(b,1)} \right] &= \tilde{J}_{(ab,1)}, \\ \left[ \tilde{J}_{(ab,0)}, \tilde{P}_{(c,1)} \right] &= f_{ab,c}{}^d \tilde{P}_{(d,1)}, & \left[ \tilde{P}_{(a,1)}, \tilde{P}_{(b,1)} \right] &= -\tilde{J}_{(ab,0)}, \end{aligned}$$

The identification  $J_{(ab,0)} \equiv J_{ab}$ ,  $J_{(ab,1)} \equiv Z_{ab}$ ,  $P_{(a,0)} \equiv P_a$  and  $P_{(a,1)} \equiv Z_a$ , leads

$$\begin{aligned} [J_{ab}, J_{cd}] &= f_{ab,cd}{}^{ef} J_{ef}, & [Z_{ab}, P_c] &= f_{ab,c}{}^d Z_d, \\ [J_{ab}, Z_{cd}] &= f_{ab,cd}{}^{ef} Z_{ef}, & [Z_{ab}, Z_c] &= -f_{ab,c}{}^d P_d, \\ [Z_{ab}, Z_{cd}] &= -f_{ab,cd}{}^{ef} J_{ef}, & [P_a, P_b] &= J_{ab}, \\ [J_{ab}, P_c] &= f_{ab,c}{}^d P_d, & [P_a, Z_b] &= Z_{ab}, \\ [J_{ab}, Z_c] &= f_{ab,c}{}^d Z_d, & [Z_a, Z_b] &= -J_{ab}, \end{aligned} \quad (61)$$

where we can see the the  $AdS_5$  algebra is a subalgebra of the  $\mathfrak{C}_5$ -algebra, i.e.,

$$AdS_5 \subset \mathfrak{C}_5. \quad (62)$$



It is interesting to note that the commutation relations of the algebra  $Z_2 \times AdS_5$  are similar to the commutation relations of the algebra  $\mathfrak{C}_5$ . The generators of the algebra  $Z_2 \times AdS_5$  satisfy the following commutation relations:

$$\begin{aligned} [J_{ab}, J_{cd}] &= f_{ab,cd}{}^{ef} J_{ef}, & [Z_{ab}, P_c] &= f_{ab,c}{}^d Z_d, \\ [J_{ab}, Z_{cd}] &= f_{ab,cd}{}^{ef} Z_{ef}, & [Z_{ab}, Z_c] &= f_{ab,c}{}^d P_d, \\ [Z_{ab}, Z_{cd}] &= f_{ab,cd}{}^{ef} J_{ef}, & [P_a, P_b] &= J_{ab}, \\ [J_{ab}, P_c] &= f_{ab,c}{}^d P_d, & [P_a, Z_b] &= Z_{ab}, \\ [J_{ab}, Z_c] &= f_{ab,c}{}^d Z_d, & [Z_a, Z_b] &= J_{ab}. \end{aligned}$$

The natural question is: what is the advantage of using the algebra  $\mathfrak{C}_5 \simeq (Z_4 \times AdS_5)_H$  instead of  $Z_2 \times AdS_5$ ? The difference between the two algebras is in invariant tensors. Indeed, the only invariant tensor algebra  $AdS_5$  is given by:

$$\langle \tilde{J}_{ab}, \tilde{J}_{cd}, \tilde{P}_e \rangle = \frac{1}{8} \varepsilon_{abcde},$$

Using the theorem 2 (see Section 5) we find that the tensor invariants for the algebra  $\mathfrak{C}_5 \simeq (Z_4 \times AdS_5)_H$  are given by

$$\begin{aligned} \langle J_{(ab,i)}, J_{(cd,j)}, P_{(e,k)} \rangle &= \alpha_\gamma K_{ijk}^\gamma \langle \tilde{J}_{ab}, \tilde{J}_{cd}, \tilde{P}_e \rangle, \\ &= \frac{1}{8} (\alpha_0 K_{ijk}^0 + \alpha_1 K_{ijk}^1 + \alpha_2 K_{ijk}^2 + \alpha_3 K_{ijk}^3) \varepsilon_{abcde}. \end{aligned} \quad (63)$$

From (98) can see that the only 3-selectors nonzero, for  $i, j, k = 0, 1$ , are  $K_{000}^0$ ,  $K_{100}^1$ ,  $K_{010}^1$ ,  $K_{001}^1$ ,  $K_{110}^2$ ,  $K_{011}^2$ ,  $K_{101}^2$ , y  $K_{111}^3$ .

On the other hand, the invariant tensors for algebra  $Z_2 \times AdS_5$  are given by (see theorem 2, section 5)

$$\begin{aligned} \langle J_{(ab,\alpha)}, J_{(cd,\beta)}, P_{(e,\gamma)} \rangle &= \alpha_\delta K_{\alpha\beta\gamma}^\delta \langle \tilde{J}_{ab}, \tilde{J}_{cd}, \tilde{P}_e \rangle, \\ &= \frac{1}{8} (\alpha_0 K_{\alpha\beta\gamma}^0 + \alpha_1 K_{\alpha\beta\gamma}^1) \varepsilon_{abcde}. \end{aligned} \quad (64)$$

From the law of multiplication of  $Z_2$  we can see that the only 3-selectors nonzero for  $\alpha, \beta, \gamma = 0, 1$ , are  $K_{000}^0$ ,  $K_{100}^1$ ,  $K_{010}^1$ ,  $K_{001}^1$ ,  $K_{110}^0$ ,  $K_{011}^0$ ,  $K_{101}^0$  and  $K_{111}^1$ .

From (63) we see that the choice  $\alpha_2 = \alpha_3 = 0$  leads to the invariant tensors for algebra  $\mathfrak{B}_5$  of [12], namely  $\langle J_{ab}, J_{cd}, P_e \rangle$ ,  $\langle Z_{ab}, J_{cd}, P_e \rangle$ ,  $\langle J_{ab}, Z_{cd}, P_e \rangle$ ,  $\langle J_{ab}, J_{cd}, Z_e \rangle$ . From (63) we see that the choice  $\alpha_1 = 0$  leads to the following invariant tensors  $\langle J_{ab}, J_{cd}, P_e \rangle$ ,  $\langle Z_{ab}, Z_{cd}, P_e \rangle$ ,  $\langle J_{ab}, Z_{cd}, Z_e \rangle$ ,  $\langle Z_{ab}, J_{cd}, Z_e \rangle$ .

It is of interest to note that the existence of four arbitrary constants in (63) allows us to choose invariant tensors that are not possible for  $Z_2 \times \mathfrak{g}$ . For example the choice of the constants

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \alpha_0 (1, -1, -1, -1), \quad (65)$$

leads to interesting Chern-Simons Lagrangian for gravitation.

From (63) we see that nonzero tensor invariants for the algebra  $\mathfrak{C}_5$  are given by (the factor  $1/8$  is absorbed in  $\alpha$ )

$$\begin{aligned}
\langle J_{ab}, J_{cd}, P_e \rangle &= \alpha_0 \varepsilon_{abcde}, \\
\langle J_{ab}, J_{cd}, Z_e \rangle &= \alpha_1 \varepsilon_{abcde}, \\
\langle J_{ab}, Z_{cd}, P_e \rangle &= \alpha_1 \varepsilon_{abcde}, \\
\langle J_{ab}, Z_{cd}, Z_e \rangle &= \alpha_2 \varepsilon_{abcde}, \\
\langle Z_{ab}, Z_{cd}, P_e \rangle &= \alpha_2 \varepsilon_{abcde}, \\
\langle Z_{ab}, Z_{cd}, Z_e \rangle &= \alpha_3 \varepsilon_{abcde}.
\end{aligned} \tag{66}$$

For simplicity it is convenient to perform the following change of basis

$$\begin{aligned}
\bar{P}_a &= \frac{1}{\sqrt{2}} P_a + \frac{1}{\sqrt{2}} Z_a, \\
\bar{Z}_a &= \frac{1}{\sqrt{2}} P_a - \frac{1}{\sqrt{2}} Z_a.
\end{aligned} \tag{67}$$

In this basis the commutation relations take the form

$$\begin{aligned}
[J_{ab}, \bar{P}_c] &= \frac{1}{\sqrt{2}} [J_{ab}, P_c] + \frac{1}{\sqrt{2}} [J_{ab}, Z_c], \\
&= \frac{1}{\sqrt{2}} f_{ab,c}^d P_d + \frac{1}{\sqrt{2}} f_{ab,c}^d Z_d, \\
&= f_{ab,c}^d \bar{P}_d,
\end{aligned}$$

$$\begin{aligned}
[J_{ab}, \bar{Z}_c] &= \frac{1}{\sqrt{2}} [J_{ab}, P_c] - \frac{1}{\sqrt{2}} [J_{ab}, Z_c], \\
&= \frac{1}{\sqrt{2}} f_{ab,c}^d P_d - \frac{1}{\sqrt{2}} f_{ab,c}^d Z_d, \\
&= f_{ab,c}^d \bar{Z}_d,
\end{aligned}$$

$$\begin{aligned}
[Z_{ab}, \bar{P}_c] &= \frac{1}{\sqrt{2}} [Z_{ab}, P_c] + \frac{1}{\sqrt{2}} [Z_{ab}, Z_c], \\
&= \frac{1}{\sqrt{2}} f_{ab,c}^d Z_d - \frac{1}{\sqrt{2}} f_{ab,c}^d P_d, \\
&= -f_{ab,c}^d \bar{Z}_d,
\end{aligned}$$

$$\begin{aligned}
[Z_{ab}, \bar{Z}_c] &= \frac{1}{\sqrt{2}} [Z_{ab}, P_c] - \frac{1}{\sqrt{2}} [Z_{ab}, Z_c], \\
&= \frac{1}{\sqrt{2}} f_{ab,c}^d Z_d + \frac{1}{\sqrt{2}} f_{ab,c}^d P_d, \\
&= f_{ab,c}^d \bar{P}_d,
\end{aligned}$$

$$\begin{aligned}
[\bar{P}_a, \bar{P}_b] &= \frac{1}{2} [P_a + Z_a, P_b + Z_b], \\
&= \frac{1}{2} ([P_a, P_b] + [P_a, Z_b] + [Z_a, P_b] + [Z_a, Z_b]), \\
&= \frac{1}{2} (J_{ab} + Z_{ab} - Z_{ba} - J_{ab}), \\
&= Z_{ab},
\end{aligned}$$

$$\begin{aligned}
[\bar{P}_a, \bar{Z}_b] &= \frac{1}{2} [P_a + Z_a, P_b - Z_b], \\
&= \frac{1}{2} ([P_a, P_b] - [P_a, Z_b] + [Z_a, P_b] - [Z_a, Z_b]), \\
&= \frac{1}{2} (J_{ab} - Z_{ab} - Z_{ba} + J_{ab}), \\
&= J_{ab},
\end{aligned}$$

$$\begin{aligned}
[\bar{Z}_a, \bar{Z}_b] &= \frac{1}{2} [P_a - Z_a, P_b - Z_b], \\
&= \frac{1}{2} ([P_a, P_b] - [P_a, Z_b] - [Z_a, P_b] + [Z_a, Z_b]), \\
&= \frac{1}{2} (J_{ab} - Z_{ab} + Z_{ba} - J_{ab}), \\
&= -Z_{ab}.
\end{aligned}$$

Moreover, the invariant tensors takes the form (the factor  $1/\sqrt{2}$  is absorbed in the coefficients  $\alpha$ )

$$\begin{aligned}
\langle J_{ab}, J_{cd}, \bar{P}_e \rangle &= \frac{1}{\sqrt{2}} \langle J_{ab}, J_{cd}, P_e \rangle + \frac{1}{\sqrt{2}} \langle J_{ab}, J_{cd}, Z_e \rangle, \\
&= (\alpha_0 + \alpha_1) \varepsilon_{abcde},
\end{aligned}$$

$$\begin{aligned}
\langle J_{ab}, J_{cd}, \bar{Z}_e \rangle &= \frac{1}{\sqrt{2}} \langle J_{ab}, J_{cd}, P_e \rangle - \frac{1}{\sqrt{2}} \langle J_{ab}, J_{cd}, Z_e \rangle, \\
&= (\alpha_0 - \alpha_1) \varepsilon_{abcde},
\end{aligned}$$

$$\begin{aligned}
\langle J_{ab}, Z_{cd}, \bar{P}_e \rangle &= \frac{1}{\sqrt{2}} \langle J_{ab}, Z_{cd}, P_e \rangle + \frac{1}{\sqrt{2}} \langle J_{ab}, Z_{cd}, Z_e \rangle, \\
&= (\alpha_1 + \alpha_2) \varepsilon_{abcde},
\end{aligned}$$

$$\begin{aligned}
\langle J_{ab}, Z_{cd}, \bar{Z}_e \rangle &= \frac{1}{\sqrt{2}} \langle J_{ab}, Z_{cd}, P_e \rangle - \frac{1}{\sqrt{2}} \langle J_{ab}, Z_{cd}, Z_e \rangle, \\
&= (\alpha_1 - \alpha_2) \varepsilon_{abcde},
\end{aligned}$$

$$\begin{aligned}\langle Z_{ab}, Z_{cd}, \bar{P}_e \rangle &= \frac{1}{\sqrt{2}} \langle Z_{ab}, Z_{cd}, P_e \rangle + \frac{1}{\sqrt{2}} \langle Z_{ab}, Z_{cd}, Z_e \rangle, \\ &= (\alpha_2 + \alpha_3) \varepsilon_{abcde},\end{aligned}$$

$$\begin{aligned}\langle Z_{ab}, Z_{cd}, \bar{Z}_e \rangle &= \frac{1}{\sqrt{2}} \langle Z_{ab}, Z_{cd}, P_e \rangle - \frac{1}{\sqrt{2}} \langle Z_{ab}, Z_{cd}, Z_e \rangle, \\ &= (\alpha_2 - \alpha_3) \varepsilon_{abcde},\end{aligned}$$

In summary, in the bases (67) the commutation relations and the invariant tensor for algebra  $\mathfrak{C}_5$  are given by

$$\begin{aligned}[J_{ab}, J_{cd}] &= f_{ab,cd}^{ef} J_{ef}, & [Z_{ab}, P_c] &= -f_{ab,c}^d Z_d, \\ [J_{ab}, Z_{cd}] &= f_{ab,cd}^{ef} Z_{ef}, & [Z_{ab}, Z_c] &= f_{ab,c}^d P_d, \\ [Z_{ab}, Z_{cd}] &= -f_{ab,cd}^{ef} J_{ef}, & [P_a, P_b] &= Z_{ab}, \\ [J_{ab}, P_c] &= f_{ab,c}^d P_d, & [P_a, Z_b] &= J_{ab}, \\ [J_{ab}, Z_c] &= f_{ab,c}^d Z_d, & [Z_a, Z_b] &= -Z_{ab},\end{aligned}\tag{68}$$

$$\begin{aligned}\langle J_{ab}, J_{cd}, P_e \rangle &= (\alpha_0 + \alpha_1) \varepsilon_{abcde}, \\ \langle J_{ab}, J_{cd}, Z_e \rangle &= (\alpha_0 - \alpha_1) \varepsilon_{abcde}, \\ \langle J_{ab}, Z_{cd}, P_e \rangle &= (\alpha_1 + \alpha_2) \varepsilon_{abcde}, \\ \langle J_{ab}, Z_{cd}, Z_e \rangle &= (\alpha_1 - \alpha_2) \varepsilon_{abcde}, \\ \langle Z_{ab}, Z_{cd}, P_e \rangle &= (\alpha_2 + \alpha_3) \varepsilon_{abcde}, \\ \langle Z_{ab}, Z_{cd}, Z_e \rangle &= (\alpha_2 - \alpha_3) \varepsilon_{abcde}.\end{aligned}\tag{69}$$

where we made the change of rotulo  $\bar{P}_a \rightarrow P_a$ ,  $\bar{Z}_a \rightarrow Z_a$

## 8 The Chern-Simons Lagrangian invariant under $\mathfrak{C}_5$

Using the subspace separation method introduced in Ref. [16] it is possible to find the Chern-Simons Lagrangian in five dimensions for the  $\mathfrak{C}_5$  algebra. Since this algebra is generated by  $J_{ab}, Z_{ab}, P_a, Z_a$  we can identify the relevant subspaces present in the  $\mathfrak{C}_5$  algebra, i.e.,

$$\mathfrak{C}_5 = \{J_{ab}\} \oplus \{Z_{ab}\} \oplus \{P_a\} \oplus \{Z_a\},$$

and write the connections in terms of pieces valued on every subspace, i.e.,

$$\begin{aligned}A &= \omega + e + k + h, \\ A_2 &= \omega + e, \\ A_1 &= \omega, \\ \bar{A} &= 0,\end{aligned}\tag{70}$$

Since  $Q^{(5)}(A, \bar{A} = 0) = Q^{(5)}(A)$ , the Chern-Simons Lagrangian is given by

$$\mathcal{L}_{\text{CS}, \mathfrak{C}_5}(A) = \kappa Q^{(5)}(A) = \kappa Q^{(5)}(A, \bar{A} = 0)\tag{71}$$

where

$$\omega = \frac{1}{2}\omega^{ab}J_{ab}, \quad k = \frac{1}{2}k^{ab}Z_{ab}, \quad e = \frac{1}{\ell}e^aP_a, \quad h = \frac{1}{\ell}h^aZ_a.$$

Repeated use of the triangle equation [16] allows us to split the Lagrangian as

$$Q^{(5)}(A, \bar{A}) = Q^{(5)}(A, A_2) + Q^{(5)}(A_2, A_1) + Q^{(5)}(A_1, \bar{A}) + dB, \quad (72)$$

where  $B$  is a 4-form given by

$$B = Q^{(4)}(A_2, A_1, \bar{A}) + Q^{(4)}(A, A_2, \bar{A}). \quad (73)$$

which correspond to a boundary term.

The calculation of the transgression forms,  $Q^{(5)}(A, A_2)$ ,  $Q^{(5)}(A_2, A_1)$  and  $Q^{(5)}(A_1, \bar{A})$ , is carried out in detail in the appendix.

The choice of arbitrary coefficients  $\alpha$  in the form (see Appendix 2, eq. (107))

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \alpha_0 (1, -1, -1, -1). \quad (74)$$

leads (modulo boundary terms) the following Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{ChS}_5, \mathfrak{C}_5} = \alpha_0 \varepsilon_{abcde} & \left( R^{ab}e^ce^de^e + \frac{3}{10\ell^2}e^ae^be^ce^de^e - \frac{3}{2}\ell^2k^{ab}R^{cd}T^e - \frac{1}{2}\ell^2R^{ab}k^{cd}k_f^ek^fh^f \right. \\ & - \frac{3}{2}k^{ab}T^ce^de^e - \frac{3}{4}\ell^2k^{ab}T^cD_\omega k^{de} - \frac{3}{2}k^{ab}T^ce^dh^e + \frac{1}{2}\ell^2k^{ab}T^ck_f^dk_f^eh^f \\ & + \frac{1}{2}k^{ab}T^ch^dh^e - \frac{1}{2}k^{ab}e^ce^dk_f^eh^f - \frac{3}{8}\ell^2k^{ab}D_\omega k^{cd}k_f^ek_f^eh^f - \frac{3}{4}k^{ab}e^ch^dk_f^eh^f \\ & + \frac{3}{10}\ell^2k^{ab}k_f^ck_f^dk_g^eh^g + \frac{3}{10}k^{ab}h^ch^dk_f^ek_f^h + \frac{3}{4}\ell^2R^{ab}R^{cd}h^e + \frac{1}{\ell^2}e^ah^be^ch^dh^e \\ & \left. + \frac{3}{20}\ell^2k_f^ak_f^bk_g^ck_g^dh^e + \frac{3}{2}R^{ab}e^ch^dh^e - \frac{3}{4}k_f^ak_f^bk_g^ck_g^dh^e - \frac{1}{2}\ell^2R^{ab}k_f^ck_f^dk^eh^e \right), \end{aligned} \quad (75)$$

where the term  $-\kappa/\ell^3$  is reabsorbed in the coefficient  $\alpha_0$ .

From (75) we can see that:

(i)  $\mathcal{L}_{\text{ChS}_5, \mathfrak{C}_5}$  contains the Einstein-Hilbert Lagrangian with a cosmological terms

(ii) for  $k^{ab} = 0$  we have

$$\mathcal{L}_{\text{ChS}_5, \mathfrak{C}_5} = \alpha_0 \varepsilon_{abcde} \left( R^{ab}e^ce^de^e + \frac{3}{10\ell^2}e^ae^be^ce^de^e + \frac{3}{4}\ell^2R^{ab}R^{cd}h^e + \frac{1}{\ell^2}e^ah^be^ch^dh^e + \frac{3}{2}R^{ab}e^ch^dh^e \right), \quad (76)$$

from where we can see that, in the particular case that  $e^a = h^a$ ,  $\mathcal{L}_{\text{ChS}_5, \mathfrak{C}_5}$  takes the form

$$\mathcal{L}_{\text{ChS}_5, \mathfrak{C}_5} = \frac{\alpha_0}{2} \varepsilon_{abcde} \left( 5R^{ab}e^ce^de^e + \frac{13}{5\ell^2}e^ae^be^ce^de^e + \frac{3}{2}\ell^2R^{ab}R^{cd}e^e \right)$$

that in the case that  $\ell \ll 1$  we have the Einstein-Hilbert Lagrangian with cosmological term.

(iii) for  $h^a = 0$  we find

$$\begin{aligned} \mathcal{L}_{\text{CS}_5, \mathfrak{C}_5} = \alpha_0 \varepsilon_{abcde} \left( R^{ab} e^c e^d e^e + \frac{3}{10\ell^2} e^a e^b e^c e^d e^e - \frac{3}{2} \ell^2 k^{ab} R^{cd} T^e - \frac{3}{2} k^{ab} T^c e^d e^e - \frac{3}{4} \ell^2 k^{ab} T^c D_\omega k^{de} \right. \\ \left. + \frac{1}{2} \ell^2 k^{ab} T^c k_f^d k^{fe} \right). \end{aligned} \quad (77)$$

from where we see that if we impose the torsion free condition, then we obtain the Einstein-Hilbert Lagrangian with cosmological term.

(iv) When  $k^{ab} = 0$  and  $h^a = 0$  we have the Einstein-Hilbert Lagrangian with a cosmological term.

## 9 (2 + 1)-dimensional case

The  $\mathfrak{C}_5 = (Z_4 \times AdS_5)_H$  algebra can be generalized to the case of arbitrary dimensions, i.e., it can be generalized to the case

$$\mathfrak{C}_d = (Z_4 \times \mathfrak{ad}\mathfrak{s}_d)_H, \quad (78)$$

whose generators satisfy the following commutation relations

$$\begin{aligned} [J_{ab}, J_{cd}] &= f_{ab,cd}^{ef} J_{ef}, & [Z_{ab}, P_c] &= f_{ab,c}^d Z_d, \\ [J_{ab}, Z_{cd}] &= f_{ab,cd}^{ef} Z_{ef}, & [Z_{ab}, Z_c] &= -f_{ab,c}^d P_d, \\ [Z_{ab}, Z_{cd}] &= -f_{ab,cd}^{ef} J_{ef}, & [P_a, P_b] &= J_{ab}, \\ [J_{ab}, P_c] &= f_{ab,c}^d P_d, & [P_a, Z_b] &= Z_{ab}, \\ [J_{ab}, Z_c] &= f_{ab,c}^d Z_d, & [Z_a, Z_b] &= -J_{ab}. \end{aligned} \quad (79)$$

The only nonzero invariant tensor for the  $AdS_3 = \mathfrak{so}(2, 2)$  algebra is given by

$$\langle J_{ab} P_c \rangle = \frac{1}{4} \varepsilon_{abc}$$

Using the theorem of Section 5 is found that

$$\langle J_{(ab,i)}, P_{(c,j)} \rangle = \frac{1}{2} (\alpha_0 K_{ij}^0 + \alpha_1 K_{ij}^1 + \alpha_2 K_{ij}^2 + \alpha_3 K_{ij}^3) \varepsilon_{abc}, \quad (80)$$

are the invariant tensors for algebra  $\mathfrak{C}_3$ . For  $Z_4$  group, the only two-selectors non-zero, with  $i, j = 0, 1$  are  $K_{00}^0$ ,  $K_{01}^1$ ,  $K_{10}^1$  y  $K_{11}^2$ . So that the corresponding invariant tensors are given by

$$\begin{aligned} \langle J_{ab}, P_c \rangle &= \alpha_0 \varepsilon_{abc}, \\ \langle J_{ab}, Z_c \rangle &= \alpha_1 \varepsilon_{abc}, \\ \langle Z_{ab}, P_c \rangle &= \alpha_1 \varepsilon_{abc}, \\ \langle Z_{ab}, Z_c \rangle &= \alpha_2 \varepsilon_{abc}. \end{aligned} \quad (81)$$

Under the change of basis

$$P_a \rightarrow \frac{1}{\sqrt{2}}P_a + \frac{1}{\sqrt{2}}Z_a, \quad (82)$$

$$Z_a \rightarrow \frac{1}{\sqrt{2}}P_a - \frac{1}{\sqrt{2}}Z_a, \quad (83)$$

the commutation relations take the form

$$\begin{aligned} [J_{ab}, J_{cd}] &= f_{ab,cd}^{ef} J_{ef}, & [Z_{ab}, P_c] &= -f_{ab,c}^d Z_d, \\ [J_{ab}, Z_{cd}] &= f_{ab,cd}^{ef} Z_{ef}, & [Z_{ab}, Z_c] &= f_{ab,c}^d P_d, \\ [Z_{ab}, Z_{cd}] &= -f_{ab,cd}^{ef} J_{ef}, & [P_a, P_b] &= Z_{ab}, \\ [J_{ab}, P_c] &= f_{ab,c}^d P_d, & [P_a, Z_b] &= J_{ab}, \\ [J_{ab}, Z_c] &= f_{ab,c}^d Z_d, & [Z_a, Z_b] &= -Z_{ab}. \end{aligned} \quad (84)$$

and corresponding invariant tensors are:

$$\begin{aligned} \langle J_{ab}, P_c \rangle &= (\alpha_0 + \alpha_1) \varepsilon_{abc}, \\ \langle J_{ab}, Z_c \rangle &= (\alpha_0 - \alpha_1) \varepsilon_{abc}, \\ \langle Z_{ab}, P_c \rangle &= (\alpha_1 + \alpha_2) \varepsilon_{abc}, \\ \langle Z_{ab}, Z_c \rangle &= (\alpha_1 - \alpha_2) \varepsilon_{abc}. \end{aligned} \quad (85)$$

For the Chern-Simons Lagrangian

$$\mathcal{L}_{\text{CS}_3, \epsilon_3} = \kappa Q^{(3)}(A, 0) = \kappa Q^{(3)}(A), \quad (86)$$

we use the method of separation of subspaces. Following the same procedure used in case 5-dimensional we find

$$\begin{aligned} \mathcal{L}_{\text{CS}_3, \epsilon_3} &= \varepsilon_{abc} \left( (\alpha_0 + \alpha_1) R^{ab} e^c + \frac{1}{3\ell^2} (\alpha_1 + \alpha_2) e^a e^b e^c + (\alpha_1 + \alpha_2) k^{ab} T^c \right. \\ &\quad + (\alpha_1 - \alpha_2) \frac{1}{2} k^{ab} D_\omega h^c - \frac{1}{2} (\alpha_1 - \alpha_2) k^{ab} k_d^c e^d + \frac{1}{3} (\alpha_1 + \alpha_2) k^{ab} k_d^c h^d \\ &\quad + (\alpha_0 - \alpha_1) R^{ab} h^c + \frac{1}{\ell^2} (\alpha_1 - \alpha_2) e^a e^b h^c + \frac{1}{2} (\alpha_1 - \alpha_2) D_\omega k^{ab} h^c \\ &\quad \left. + \frac{1}{\ell^2} (\alpha_0 - \alpha_1) e^a h^b h^c - \frac{1}{3} (\alpha_0 - \alpha_1) k_d^a k^{db} h^c - \frac{1}{3\ell^2} (\alpha_1 - \alpha_2) h^a h^b h^c \right). \end{aligned} \quad (87)$$

From (87) we can see that:

(i) When  $k^{ab} = 0$  and  $h^a = 0$  we have the Einstein-Hilbert Lagrangian with cosmological term for certain values of the constant  $\alpha$ .

The arbitrariness of the constant  $\alpha$  in terms of Einstein-Hilbert and the cosmological constant allow interpreting the Lagrangian as the Lanczos-Lovelock Lagrangian in 3 dimensions

This occurs even in cases where  $\alpha_0 - \alpha_1 = 0$ ,  $\alpha_1 = 0$  y  $\alpha_1 - \alpha_2 = 0$ . This result allows us to interpret the Lagrangian (87) as a Lagrangian describing a link between Lovelock gravity and bosonic fields  $k^{ab}$  and  $h^a$ .

(ii) In the case that  $h^a = 0$  we have

$$\begin{aligned} \mathcal{L}_{\text{CS}_3, C_3} &= \varepsilon_{abc} \left( (\alpha_0 + \alpha_1) R^{ab} e^c + \frac{1}{3\ell^2} (\alpha_1 + \alpha_2) e^a e^b e^c + (\alpha_1 + \alpha_2) k^{ab} T^c - \frac{1}{2} (\alpha_1 - \alpha_2) k^{ab} k_d^c e^d \right). \end{aligned} \quad (88)$$

Note that if  $T^a = 0$  and  $\alpha_1 = \alpha_2$ , then we obtain the Einstein-Hilbert Lagrangian with cosmological term.

## 10 Comments

In the present article we have introduced a modification to the Lie algebra  $S$ -expansion procedure. The modification is carried out by imposing a condition, which was called  $H$ -condition, on the  $S$ -expansion procedure, when the semigroup is given by a cyclic group of even order. The invariant tensors for  $S_H$ -expanded algebras are calculated and the dual formulation of  $S_H$ -expansion procedure was found.

Following Ref. [12], we have considered the  $S_H$ -expansion of the five-dimensional  $AdS$  algebra and its corresponding invariants tensors were obtained. Then a Chern-Simons Lagrangian invariant under the five-dimensional  $AdS$  algebra  $S_H$ -expanded is constructed and its relationship to the general relativity was studied.

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## 11 Appendix 1: Cyclic groups

### 11.1 Semigroups

A semigroup is a closed algebraic structure which also satisfies the axiom of associativity:

$$\forall a, b, c \in A, \quad a * (b * c) = (a * b) * c. \quad (89)$$

For an associative structure no matter in what manner we perform the partial products of an expression. Therefore it is possible to define a  $r$ -selector to a semigroup of  $|S|$  elements:

$$\lambda_{\alpha_1} \cdots \lambda_{\alpha_r} = \lambda_{\rho(\alpha_1, \dots, \alpha_r)} = K_{\alpha_1 \dots \alpha_r}^\gamma \lambda_\gamma. \quad (90)$$

For an abelian semigroup, we have  $\lambda_\alpha \lambda_\beta = \lambda_\beta \lambda_\alpha$  so that  $\lambda_{\rho(\alpha, \beta)} = \lambda_{\rho(\beta, \alpha)}$ . This mean that

$$K_{\alpha\beta}^\gamma = K_{\beta\alpha}^\gamma. \quad (91)$$



In general  $K_{\alpha_1 \dots \alpha_r}^\gamma$  is completely symmetric in their lower indices. Consequently the useful identity is true:

$$K_{\alpha\beta\gamma}^\delta = K_{\beta\gamma\alpha}^\delta = K_{\gamma\alpha\beta}^\delta. \quad (92)$$

Since

$$\begin{aligned} \lambda_\alpha \lambda_\beta &= K_{\alpha\beta}^\gamma \lambda_\gamma, \\ \lambda_\alpha \lambda_\beta \lambda_\delta &= K_{\alpha\beta}^\gamma \lambda_\gamma \lambda_\delta, \\ K_{\alpha\beta\delta}^\varepsilon \lambda_\varepsilon &= K_{\alpha\beta}^\gamma K_{\gamma\delta}^\varepsilon \lambda_\varepsilon, \end{aligned}$$

we have

$$K_{\alpha\beta\gamma}^\delta = K_{\alpha\beta}^\varepsilon K_{\varepsilon\gamma}^\delta \quad (93)$$

and in general

$$K_{\alpha_1 \dots \alpha_r}^\gamma = K_{\alpha_1 \alpha_2}^{\beta_1} K_{\beta_1 \alpha_3}^{\beta_2} \dots K_{\beta_{r-3} \alpha_{r-1}}^{\beta_{r-2}} K_{\beta_{r-2} \alpha_r}^\gamma. \quad (94)$$

## 11.2 Direct product of semigroups

Let  $S$  and  $S'$  be two semigroups. The Cartesian product  $S \times S'$  together with the internal binary operation

$$(a, a')(b, b') = (ab, a'b') \quad (95)$$

form a semigroup. Indeed, the closure property is satisfied by construction (since  $ab \in S$  and  $a'b' \in S'$ ). Let's check the associative property:

$$\begin{aligned} (a, a')[(b, b')(c, c')] &= (a, a')(bc, b'c'), \\ &= (a(bc), a'(b'c')), \\ &= ((ab)c, (a'b')c'), \\ &= (ab, a'b')(c, c'), \\ &= [(a, a')(b, b')(c, c')]. \end{aligned}$$

This means that the associative property is satisfied, as a direct consequence from the associativity of  $S$  and  $S'$ . In the representation of selectors, we denote the internal law of the semigroup  $S \times S' = \{\lambda_\alpha \times \lambda'_{\alpha'}\}$  as

$$(\lambda_\alpha \times \lambda'_{\alpha'}) (\lambda_\beta \times \lambda'_{\beta'}) = (\lambda_\alpha \lambda_\beta) \times (\lambda'_{\alpha'} \lambda'_{\beta'}) = K_{\alpha\beta}^\gamma K'_{\alpha'\beta'}^{\gamma'} \lambda_\gamma \times \lambda'_{\gamma'}. \quad (96)$$

## 11.3 Cyclic groups

Cyclic groups  $\mathbb{Z}_n$  are abelian groups whose elements can be expressed as a power of a single element of the group. That is, if  $a \in \mathbb{Z}_n$ , then

$$\mathbb{Z}_n = \{a, aa, aaa, \dots, a^n\}. \quad (97)$$

The group  $\mathbb{Z}_4$  has the Cayley table

$Z_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

or in the notation of selectors

$Z_4$	$\lambda_0$	$\lambda_1$	$\lambda_2$	$\lambda_3$
$\lambda_0$	$\lambda_0$	$\lambda_1$	$\lambda_2$	$\lambda_3$
$\lambda_1$	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_0$
$\lambda_2$	$\lambda_2$	$\lambda_3$	$\lambda_0$	$\lambda_1$
$\lambda_3$	$\lambda_3$	$\lambda_0$	$\lambda_1$	$\lambda_2$

(98)

Some properties for cyclic groups of even order,  $Z_{2n}$ , are

$$\begin{aligned}
K_{k+n,l}^m &= \begin{cases} 1, & m \equiv k+n+l \pmod{2n} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & m+n \equiv k+2n+l \pmod{2n} \\ 0, & \text{otherwise} \end{cases} \\
&= K_{kl}^{m+n}.
\end{aligned}
\tag{99}$$

$$\begin{aligned}
K_{k+n,l}^m &= \begin{cases} 1, & m+n \equiv k+n+l \pmod{2n} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & m+2n \equiv k+2n+l \pmod{2n} \\ 0, & \text{otherwise} \end{cases} \\
&= K_{kl}^m.
\end{aligned}
\tag{100}$$

$$\begin{aligned}
K_{i+n,j+n}^\gamma &= \begin{cases} 1, & \gamma \equiv i+n+j+n \pmod{2n} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \gamma \equiv i+j+2n \pmod{2n} \\ 0, & \text{otherwise} \end{cases} \\
&= K_{ij}^\gamma.
\end{aligned}
\tag{101}$$

$$\begin{aligned}
K_{i,j+n}^k &= \begin{cases} 1, & k \equiv i+j+n \pmod{2n} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & k \equiv (i+n)+j \pmod{2n} \\ 0, & \text{otherwise} \end{cases} \\
&= K_{i+n,j}^k.
\end{aligned}
\tag{102}$$

## 11.4 The Klein group

It is possible to show that there are only two groups of four elements (up to isomorphism): one is the cyclic group  $Z_4$ , and the other is the group of Klein  $D_4$ . The Klein group  $D_4$  corresponds to the direct product  $Z_2 \times Z_2$ . To prove this consider the multiplication rule of the direct product of (semi) groups. If we denote the factor groups as  $(Z_2, \bar{\cdot})$  and as  $(Z_2, \tilde{\cdot})$ ,

$Z_2$	$\bar{0}$	$\bar{1}$
$\bar{0}$	$\bar{0}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	$\bar{0}$

$Z_2$	$\tilde{0}$	$\tilde{1}$
$\tilde{0}$	$\tilde{0}$	$\tilde{1}$
$\tilde{1}$	$\tilde{1}$	$\tilde{0}$

we have

$Z_2 \times Z_2$	$(\bar{0}, \bar{0})$	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{1})$
$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{0})$	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{1})$
$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{0})$	$(\bar{1}, \bar{1})$	$(\bar{0}, \bar{1})$
$(\bar{0}, \bar{1})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{1})$	$(\bar{0}, \bar{0})$	$(\bar{1}, \bar{0})$
$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{1})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{0})$

Denoting by  $\lambda_0 = (\bar{0}, \bar{0})$ ,  $\lambda_1 = (\bar{1}, \bar{0})$ ,  $\lambda_2 = (\bar{0}, \bar{1})$  y  $\lambda_3 = (\bar{1}, \bar{1})$  we see that the multiplication table of the Klein group, is given by

$D_4$	$\lambda_0$	$\lambda_1$	$\lambda_2$	$\lambda_3$
$\lambda_0$	$\lambda_0$	$\lambda_1$	$\lambda_2$	$\lambda_3$
$\lambda_1$	$\lambda_1$	$\lambda_0$	$\lambda_3$	$\lambda_2$
$\lambda_2$	$\lambda_2$	$\lambda_3$	$\lambda_0$	$\lambda_1$
$\lambda_3$	$\lambda_3$	$\lambda_2$	$\lambda_1$	$\lambda_0$

(103)

## 12 Appendix 2: Transgression forms

### 12.1 Calculation of $Q^{(5)}(A_1, \bar{A})$

From equations (70), we have

$$A_t(A_1, \bar{A}) = \bar{A} + t(A_1 - \bar{A}) = t\omega,$$

so that

$$\begin{aligned} F_t(A_1, \bar{A}) &= dA_t(A_1, \bar{A}) + \frac{1}{2} [A_t(A_1, \bar{A}), A_t(A_1, \bar{A})], \\ &= t d\omega + \frac{1}{2} [t\omega, t\omega], \\ &= t d\omega + \frac{t^2}{2} [\omega, \omega], \end{aligned}$$

with

$$\begin{aligned} d\omega &= \left( \frac{1}{2} \omega^{ab} J_{ab} \right) \propto J_{ab} \\ [\omega, \omega] &= \frac{1}{4} \omega^{ab} \omega^{cd} [J_{ab}, J_{cd}] \propto J_{ab}. \end{aligned}$$

Since there are no invariant tensors of the form  $\langle J_{ab}, J_{cd}, J_{ef} \rangle$ , we have

$$Q^{(5)}(A_1, \bar{A}) = 3 \int_0^1 dt \left\langle \omega \left( t d\omega + \frac{t^2}{2} [\omega, \omega] \right)^2 \right\rangle = 0.$$

## 12.2 Calculation of $Q^{(5)}(A_2, A_1)$

Similarly to the previous case, we have

$$A_t(A_2, A_1) = A_1 + t(A_2 - A_1) = \omega + te,$$

so that

$$\begin{aligned} F_t(A_2, A_1) &= dA_t(A_2, A_1) + \frac{1}{2} [A_t(A_2, A_1), A_t(A_2, A_1)], \\ &= d\omega + tde + \frac{1}{2} [\omega + te, \omega + te], \\ &= d\omega + tde + \frac{1}{2} [\omega, \omega] + \frac{t}{2} [\omega, e] + \frac{t}{2} [e, \omega] + \frac{t^2}{2} [e, e], \end{aligned} \quad (104)$$

from where

$$\begin{aligned} R &= d\omega + \frac{1}{2} [\omega, \omega] \propto J_{ab}, \\ T &= de + [\omega, e] \propto P_a, \\ [e, e] &= \frac{1}{\ell^2} e^a e^b [P_a, P_b] \propto Z_{ab}. \end{aligned}$$

and therefore

$$\begin{aligned} Q^{(5)}(A_2, A_1) &= 3 \int_0^1 dt \left\langle e \left( R + tT + \frac{t^2}{2} [e, e] \right)^2 \right\rangle, \\ &= 3 \int_0^1 dt \left\langle e \left( R + \frac{t^2}{2} [e, e] \right)^2 \right\rangle, \\ Q^{(5)}(A_2, A_1) &= 3 \int_0^1 dt \left\langle R^2 e + t^2 R [e, e] e + \frac{t^4}{4} [e, e]^2 e \right\rangle \end{aligned}$$

so that

$$\begin{aligned}
& Q^{(5)}(A_2, A_1) \\
&= 3 \left\langle R^2 e + \frac{1}{3} R[e, e] e + \frac{1}{20} [e, e]^2 e \right\rangle, \\
&= 3 \left( \frac{1}{4\ell} R^{ab} R^{cd} e^e \langle J_{ab}, J_{cd}, P_e \rangle + \frac{1}{6\ell^3} R^{ab} e^c e^d e^e \langle J_{ab}, [P_c, P_d], P_e \rangle \right. \\
&\quad \left. + \frac{1}{20\ell^5} e^a e^b e^c e^d e^e \langle [P_a, P_b], [P_c, P_d], P_e \rangle \right), \\
&= 3 \left( \frac{1}{4\ell} R^{ab} R^{cd} e^e \langle J_{ab}, J_{cd}, P_e \rangle + \frac{1}{6\ell^3} R^{ab} e^c e^d e^e \langle J_{ab}, Z_{cd}, P_e \rangle + \frac{1}{20\ell^5} e^a e^b e^c e^d e^e \langle Z_{ab}, Z_{cd}, P_e \rangle \right), \\
&= 3 \varepsilon_{abcde} \left( \frac{1}{4\ell} (\alpha_0 + \alpha_1) R^{ab} R^{cd} e^e + \frac{1}{6\ell^3} (\alpha_1 + \alpha_2) R^{ab} e^c e^d e^e + \frac{1}{20\ell^5} (\alpha_2 + \alpha_3) e^a e^b e^c e^d e^e \right). \tag{105}
\end{aligned}$$

If we demand that the constants  $\alpha_1, \alpha_2$  and  $\alpha_3$  satisfy the conditions

$$\begin{aligned}
\alpha_0 + \alpha_1 &= 0, \\
\alpha_1 - \alpha_2 &= 0, \\
\alpha_2 - \alpha_3 &= 0,
\end{aligned} \tag{106}$$

we find

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \alpha_0 (1, -1, -1, -1). \tag{107}$$

In this case the invariant tensors (69) are given by

$$\begin{aligned}
\langle J_{ab}, J_{cd}, Z_e \rangle &= \alpha_0 \varepsilon_{abcde}, \\
\langle J_{ab}, Z_{cd}, P_e \rangle &= -\alpha_0 \varepsilon_{abcde}, \\
\langle Z_{ab}, Z_{cd}, P_e \rangle &= -\alpha_0 \varepsilon_{abcde}.
\end{aligned} \tag{108}$$

and  $Q^{(5)}(A_2, A_1)$  takes the form

$$Q^{(5)}(A_2, A_1) = 3 \varepsilon_{abcde} \left( -\frac{1}{3\ell^3} \alpha_0 R^{ab} e^c e^d e^e - \frac{1}{10\ell^5} \alpha_0 e^a e^b e^c e^d e^e \right). \tag{109}$$

Note that the form of transgression (105) contains the same terms as the Lagrangian of Lovelock on 5 dimensions for any value of the coefficients  $\alpha$ . We must not confuse the coefficients  $\alpha$  with the coefficients of Lovelock. If we denote by  $\beta$  the coefficients of Lovelock, then the relationship between the expansion coefficients  $\alpha$  and the coefficients of Lovelock, is given by

$$\begin{aligned}
\beta_0 &= (\alpha_0 + \alpha_1) / 2, \\
\beta_1 &= (\alpha_1 + \alpha_2) / (3\ell^2), \\
\beta_2 &= (\alpha_2 + \alpha_3) / (10\ell^4).
\end{aligned} \tag{110}$$

### 12.3 Calculation of $Q^{(5)}(A, A_2)$

In this case we have

$$A_t(A, A_2) = A_2 + t(A - A_2) = \omega + e + t(k + h),$$

so that

$$\begin{aligned} F_t(A, A_2) &= d\omega + de + t(dk + dh) + \frac{1}{2}[\omega, \omega] + [\omega, e] + t[\omega, k] + t[\omega, h] \\ &+ \frac{1}{2}[e, e] + t[e, h] + t[k, e] + \frac{t^2}{2}[k, k] + t^2[k, h] + \frac{t^2}{2}[h, h], \\ &= d\omega + \frac{1}{2}[\omega, \omega] + de + [\omega, e] + \frac{1}{2}[e, e] + t(dk + [\omega, k] + dh + [\omega, h] + [k, e] + [e, h]) \\ &+ t^2\left(\frac{1}{2}[k, k] + \frac{1}{2}[h, h] + [k, h]\right), \\ &= R + T + \frac{1}{2}[e, e] + t(D_\omega k + D_\omega h + [k, e] + [e, h]) + \frac{t^2}{2}([k, k] + [h, h] + 2[k, h]), \end{aligned}$$

from where

$$\begin{aligned} R &= d\omega + \frac{1}{2}[\omega, \omega] \propto J_{ab}, & [k, e] &\propto Z_a, \\ T &= de + [\omega, e] \propto P_a, & [e, h] &\propto J_{ab}, \\ [e, e] &\propto Z_{ab}, & [k, k] &\propto J_{ab}, \\ D_\omega k &= dk + [\omega, k] \propto Z_{ab}, & [h, h] &\propto Z_{ab}, \\ D_\omega h &= dh + [\omega, h] \propto Z_a, & [k, h] &\propto P_a, \end{aligned}$$

and therefore

$$\begin{aligned} Q^{(5)}(A, A_2) &= 3\alpha_0 \varepsilon_{abcde} \left( \frac{1}{2\ell} k^{ab} R^{cd} T^e + \frac{1}{6\ell} R^{ab} k^{cd} k_f^e h^f + \frac{1}{2\ell^3} k^{ab} T^c e^d e^e + \frac{1}{4\ell} k^{ab} T^c D_\omega k^{de} \right. \\ &+ \frac{1}{2\ell^3} k^{ab} T^c e^d h^e - \frac{1}{6\ell} k^{ab} T^c k_f^d k^e h^f - \frac{1}{6\ell^3} k^{ab} T^c h^d h^e + \frac{1}{6\ell^3} k^{ab} e^c e^d k_f^e h^f \\ &+ \frac{1}{8\ell} k^{ab} D_\omega k^{cd} k_f^e h^f + \frac{1}{4\ell^3} k^{ab} e^c h^d k_f^e h^f - \frac{1}{10\ell} k^{ab} k_f^c k^d k_g^e h^g - \frac{1}{10\ell^3} k^{ab} h^c h^d k_f^e k^f \\ &- \frac{1}{4\ell} R^{ab} R^{cd} h^e - \frac{1}{3\ell^5} e^a h^b e^c h^d h^e - \frac{1}{20\ell} k_f^a k^b k_g^c k^d h^e - \frac{1}{2\ell^3} R^{ab} e^c h^d h^e \\ &\left. + \frac{1}{4\ell^3} k_f^a k^b k^c h^d h^e + \frac{1}{6\ell} R^{ab} k_f^c k^d h^e \right). \end{aligned}$$

### References

- [1] M. Hatsuda, M. Sakaguchi, Prog. Theor. Phys. **109** (2003) 853. arXiv: hep-th/0106114.
- [2] J. A. de Azcárraga, J. M. Izquierdo, M. Picón, O. Varela, Nucl. Phys. B **662** (2003) 185. arXiv: hep-th/0212347.

- [3] J. A. de Azcárraga, J. M. Izquierdo, M. Picón, O. Varela, *Class. Quant. Grav.* **21** (2004) S1375. arXiv: hep-th/0401033.
- [4] J. A. de Azcárraga, J. M. Izquierdo, M. Picón, O. Varela, *Int. J. Theor. Phys.* **46** (2007) 2738. arXiv: hep-th/0703017.
- [5] E. İnönü, E. P. Wigner, *Proceedings of the National Academy of Sciences of the United States of America*, **39(6)** (1953) 510.
- [6] E. Weimar-Woods, *Jour. Math. Phys.*, **32** (1991) 2028.
- [7] E. Weimar-Woods, *Jour. Math. Phys.*, **32** (1991) 2660.
- [8] E. Weimar-Woods, *Rev. Math. Phys.*, **12(11)** (2000) 1505-1529.
- [9] F. Izaurieta, E. Rodríguez, P. Salgado, *J. Math. Phys.* **47** (2006) 123512. arXiv: hep-th/0606215.
- [10] F. Izaurieta, E. Rodríguez, P. Salgado, *J. Phys. Conf. Ser.* **134** (2008) 012005.
- [11] F. Izaurieta, A. Perez, E. Rodriguez, P. Salgado, *Jour. Math. Phys.* **50** (2009) 073511.
- [12] F. Izaurieta, P. Minning, A. Pérez, E. Rodriguez, P. Salgado, *Phys. Lett. B*, **678(2)** (2009) 213-217. arXiv:0905.2187v2 [hep-th]
- [13] A.H. Chamseddine, *Phys. Lett. B* **233** (1989) 291
- [14] A.H. Chamseddine, *Nucl. Phys. B* **346** (1990) 213
- [15] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008)*. (2005). arXiv:hep-th/0502193
- [16] F. Izaurieta, E. Rodríguez, P. Salgado, *Lett. Math. Phys.*, **80(2)** (2007) 127-138. arXiv:hep-th/0603061v2